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A MALCEV-ADMISSIBLE MUTATION OF AN ALTERNATIVE ALGEBRA

In memory of Professor Dock Sang Rim

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1. Introduction

Our investigation stems from the (r,s)-mutation of an associative algebra which originates from Santilli's generalization of classical and quantum mechanics. Santilli has introduced a time evolution law

$$\frac{dx}{dt}=i(xrh-hsx) \tag{1}$$

as a generalization of the conventional Heisenberg equation, where h is a Hamiltonian and r, s are fixed invertible operators on a Hilbert space. Let A be an associative algebra with multiplication xy. The right side of the equation (1) leads to introduce a genuine nonassociative product

$$x*y=xry-ysx \tag{2}$$

on the vector space A. The resulting algebra has been called the *(r,s)-mutation* of A and denoted by $A(r,s)$.

The (r,s)-mutation $A(r,s)$ of an associative algebra is Lie-admissible in the sense that the algebra $A(r,s)^-$ with the commutator product $[x,y]^*=x*y-y*x$ is a Lie algebra, that is, $A(r,s)^-$ satisfies the Jacobi identity

$$[[x,y]^*,z]^*+[[y,z]^*,x]^*+[[z,x]^*,y]^*=0. \tag{3}$$

The structure of $A(r,s)$ has been investigated by a number of authors [5, 6, 7, 8].

In this paper we extend the (r,s)-mutation of an associative algebra to an alternative algebra A and prove some basic structure theorems for $A(r,s)$. An algebra A is called *alternative* if it satisfies a weak associativity

$$x(xy)=x^2y \text{ and } yx^2=(yx)x$$

for all $x,y\in A$. A Cayley-Dickson algebra is a well known alternative algebra that is not associative. Let a be a fixed element of an alternative algebra A with product xy. Define a product $x\cdot y$ by

$$x\cdot y=(xa)y$$

on the vector space A. The resulting algebra, denoted by $A^{(a)}$, is called the (left) a-*homotope* of A[3]. A right a-homotope is similarly defined. If A has a unity 1 and a is invertible, $A^{(a)}$ is called the (left) *a-isotope* of A. It is shown [3] that if A is alternative then every homotope of A is alternative also. Following the relation (2), we define the (left) (r, s)-*mutation* $A(r, s)$ of A as the algebra with product

$$x*y=(xr)y-(ys)x \tag{4}$$

defined on the vector space A, where r and s are fixed elements in A. Note that if A is associative then (4) reduces to (2).

The (r, s)-mutation $A(r, s)$ of an alternative algebra is not in genreal Lie-admissible but Malcev-admissible in the sense that the minus algebra $A(r, s)^-$ with product $[x, y]^*=x*y-y*x$ is a Malcev algebra, namely, $A(r, s)^-$ satisfies the Malcev identity

$$[[x, y]^*, [x, z]^*]^*=[[[x, y]^*, z]^*, x]^*+[[[y, z]^*, x]^*, x]^*+[[[z, x]^*, x]^*, y]^*. \tag{5}$$

Since any Lie algebra is a Malcev algebra, the (r, s)-mutation $A(r, s)$ can be utilized for a possible generalization of Santilli's mechanics as well as octonionic mechanics.

2. A generalization of the (r, s)-mutation

Following Myung [4], an algebra B is termed *Malcev-admissible* if the minus algebra B^- is a Malcev algebra, that is, the commutator $[x, y]=xy-yx$ satisfies the identity (5). It is well known that any Lie-admissible and alternative algebra are Malcev-admissible and that a Cayley-Dickson algebra of characteristic$\neq 3$ is Malcev-admissible but not Lie-admissible (see [4]).

Let A be a vector space over a field F and let f be a bilinear mapping: $A\times A\to A$. Denote by $A(f)$ the algebra with multiplication $f(x, y)$ defined on the vector space A. We can also define a skew symmetric bilinear mapping $f^-: A\times A\to A$ by

$$f^-(x, y)=f(x, y)-f(y, x).$$

Clearly, $A(f)^-=A(f^-)$. If the algebra $A(f)$ is Lie-admissible or Malcev-admissible then we call f Lie-admissible or Malcev-admissible. Let f and g be fixed bilinear mappings : $A\times A\to A$. Using f and g, we define a new product $x\circ y$ on the vector space A by

$$x\circ y=f(x, y)-g(y, x). \tag{6}$$

Denote by $A_{f,g}$ the algebra with product $x\circ y$. The (r, s)-mutation $A(r, s)$ defined by (4) is a special case of $A_{f,g}$ where $f(x, y)=(xr)y$ and $g(x, y)=(xs)y$.

Let f and g be Malcev-admissible. We give a condition that the algebra $A_{f,g}$ is Malcev-admissible in terms of a 2-cocycle. Let h and k be bilinear mappings : $A\times A\to A$. We define quadra-linear mappings $k\triangle h$ and $k\square h$: $A\times A\times A\times A\to A$ by

$$(k\triangle h)(x, y, z, t)=k(k(h(x, y), z), t)+k(h(k(x, y), z), t)+h(k(k(x, y), z), t), \tag{7}$$

$$(k\square h)(x, y, z, t)=k(h(x, y), k(z, t))+k(k(x, y), h(z, t))+h(k(x, y), k(z, t)) \tag{8}$$

for $x, y, z, t\in A$.

If M is a Malcev algebra with product $[x, y]$, then following general bimodule theory ([2, p. 93]), a skew-symmetric bilinear mapping $h: M\times M\to M$ is called a 2-*cocycle* of M if h satisfies the identity

$$(k\triangle h)(x, y, z, x)+(k\triangle h)(y, z, x, x)+(k\triangle h)(z, x, x, y)+(k\square h)(x, z, x, y)=0 \tag{9}$$

for $x, y, z\in M$, where $k(x, y)=[x, y]$. Let B be a Malcev-admissible algebra. As for a Lie-admissible algebra [5], it can be shown that a bilinear mapping $h: B\times B\to B$ is a 2-cocycle of B if and only if h^- is a 2-cocycle of B^-. We use this to give a condition that the algebra $A_{f,g}$ is Malcev-admissible for fixed Malcev-admissible bilinear mappings

f, g.

THECREM 1. *Let A be a vector space and let f, g be Malcev-admissible bilinear mappings* : $A\times A\to A$. *Then the algebra $A_{f,g}$ defined by* (6) *is Malcev-admissible if and only if f and g satisfy*

$$\{f^-, g^-\}(x, y, z, x)+\{f^-, g^-\}(y, z, x, x) + \{f^-, g^-\}(z, x, x, y)+(f^-\square g^-+g^-\square f^-)(x, z, x, y)=0 \tag{10}$$

for $x, y, z\in A$, where $\{f^-, g^-\}=f^-\triangle g^-+g^-\triangle f^-$. In particular, if f and g are 2-cocycles of $A(g)$ and $A(f)$ respectively, then $A_{f,g}$ is Malcev-admissible.

Proof. Denote $[x, y]=x\circ y-y\circ x$. Then $[x, y]=f(x, y)-g(y, x)-f(y, x)+g(x, y)=f^-(x, y)+g^-(x, y)$. Using this, we compute

$$[[[x, y], z], x]=f^-(f^-(f^-(x, y), z), x)+(f^-\triangle g^-+g^-\triangle f^-)(x, y, z, x) +g^-(g^-(g^-(x, y), z), x),$$
$$[[[y, z], x], x]=f^-(f^-(f^-(y, z), x), x)+(f^-\triangle g^-+g^-\triangle f^-)(y, z, x, x) +g^-(g^-(g^-(y, z), x), x),$$
$$[[[z, x], x], y]=f^-(f^-(f^-(z, x), x), y)+(f^-\triangle g^-+g^-\triangle f^-)(z, x, x, y) +g^-(g^-(g^-(z, x), x), y)$$
$$[[x, z], [x, y]]=f^-(f^-(x, z), f^-(x, y))+(f^-\square g^-+g^-\square f^-)(x, z, x, y) +g^-(g^-(x, z), g^-(x, y)).$$

Adding these four equations, we have that the sum of terms involving only f^- or g^- on the right sides is zero, since f and g are Malcev-admissible. The remaining terms add to the left side of (10). Therefore, the Malcev identity in $A_{f,g}{}^-$ is equivalent to the identity (10). If f and g are 2-cocycles of $A(g)$ and $A(f)$ then (9) holds for f^- and g^-, and this gives (10).

REMARK. If f and g are Lie-admissible then it can be similarly shown that $A_{f,g}$ is Lie-admissible if and only if f and g are 2-cocycles of $A(g)$ and $A(f)$, respectively. That is, $A_{f,g}$ is Lie-admissible if and only if $f^-(g^-(x, y), z)+f^-(g^-(y, z), x)+f^-(g^-(z, x), y)+g^-(f^-(x, y), z)+g^-(f^-(y, z), x)+g^-(f^-(z, x), y)=0$ holds for all $x, y, z\in A$.

3. The (r, s)-mutation of an alternative algebra

We focus on the (left) (r, s)-mutation $A(r, s)$ of an alternative algebra A. Thus the product $x*y$ in $A(r, s)$ is given by (4). Denote the associator and commutator in $A(r, s)$ by

$$(x, y, z)^*=(x*y)*z-x*(y*z),$$
$$[x, y]^*=x*y-y*x.$$

It follows from (4) that

$$[x, y]^*=(xr)y-(ys)x-(yr)x+(xs)y =[x(r+s)]y-[y(r+s)]x, \tag{11}$$

$$(x, y, z)^*=[(xr)y-(ys)x]*z-x*[(yr)z-(zs)y] =[((xr)y)r]z-[((ys)x)r]z-(zs)[(xr)y]+(zs)[(ys)x] -(xr)[(yr)z]+(xr)[(zs)y]+[((yr)z)s]x-[((zs)y)s]x. \tag{12}$$

Since the $(r+s)$-homotope $A^{(r+s)}$ is alternative and $A(r,s)^- \simeq A^{(r+s)-}$, $A(r,s)$ is Malcev-admissible. Also, the Jordan product $\{x,y\}^* = x*y+y*x$ is given by

$$\{x,y\}^* = [x(r-s)]y + [y(r-s)]x. \tag{13}$$

Hence $A(r,s)^+ \simeq A^{(r-s)+}$, where $A(r,s)^+$ denotes the algebra with product $\{x,y\}^*$ defined on the vector space $A(r,s)$. Since any alternative algebra is Jordan-admissible, this implies that $A(r,s)$ is Jordan-admissible also.

Define the commutative center $K(A)$, the nucleus $N(A)$ and the center $Z(A)$ of A by $K(A)=\{x\in A \mid [x,A]=0\}$, $N(A)=\{x\in A \mid (A,A,x)=(A,x,A)=(x,A,A)=0\}$ and $Z(A)=K(A)\cap N(A)$, where $[x,y]=xy-yx$ and $(x,y,z)=(xy)z-x(yz)$. If r and s are in $N(A)$ then the left and right (r,s)-mutations of A coincide.

As for the (r,s)-mutation of an associative algebra [5, 6, 8], we investigate other identities satisfied by $A(r,s)$ which are not consequences of Malcev-admissibility and Jordan-admissibility. Two identities which are useful in the study of nonassociative algebras are the flexible identity,

$$(x*y)*x = x*(y*x). \tag{14}$$

and the third power identity,

$$(x*x)*x = x*(x*x) \tag{15}$$

which is implied by flexibility. With the exception of Malcev-admissibility and Jordan-admissiblity, the identity (15) is implied by virtually all the identities which are considered in nonassociative algebras. In fact, Osborn [8] has shown that if A is an associative algebra of characteristic $\neq 2, 3$ and r, s are invertible then (15) in $A(r,s)$ implies most of the well known nonassociative identities. We prove the same result for an alternative algebra, when one of r and s is invertible in $N(A)$.

To state our result, we need some definitions. A nonassociative algebra B is called *power-associative* if the subalgebra of B generated by every element in B is associative. We also call B *generalized quasi-alternative* if, up to isomorphism, it arises from an alternative algebra A under the product $x*y=\alpha xy+\beta yx$ for some fixed α, β in the center $Z(A)$ of A. Thus, $B\simeq A(\alpha, -\beta)$. If α and β are just scalars then B is called *quasi-alternative*. It is easy to check that any generalized quasi-alternative algebra is both flexible and power-associative.

Let A be an alternative algebra over a field F. Recall Moufang identities in A:

$$(aba)x = a[b(ax)], \tag{16}$$
$$x(aba) = [(xa)b]a, \tag{17}$$
$$a(xy)a = (ax)(ya). \tag{18}$$

Hence, using (17) and (18), we have

$$[((xr)y)r]x = [x(ryr)]x = (xr)[(yr)x],$$
$$[((xs)y)s]x = [x(sys)]x = (xs)[(ys)x].$$

Substituting these in the relation (12) with $x=z$, we get

$$(x,y,x)^* = (xr)[(xs)y] + [((yr)x)s]x - [((ys)x)r]x - (xs)[(xr)y]. \tag{19}$$

Thus, if $s=\alpha r$ for some α in the center $Z(A)$ then (19) implies the flexible identity $(x,y,x)^*=0$ in $A(r,\alpha r)$. Also, $A(r,\alpha r)$ is power-associative and the nth power x^{*n} in

$A(r, \alpha r)$ is given by $x^{*n}=(1-\alpha)^{n-1}\, x(rx)^{n-1}$. This follows from Artin's theorem that the subalgebra of A generated by any two elements is associative. Therefore, we can state

THECREM 2. *Let r be a fixed element in an alternative algebra A and let α be in the center $Z(A)$ of A. Then $A(r, \alpha r)$ is both flexible and power-associative.*

If A is associative and r, s are invertible then it is shown in [8] that the converse of Theorem 2 is true. We prove the converse of Theorem 2 for an alternative algebra in the following theorem.

THEOREM 3. *Let A be an alternative algebra with unity 1 over a field F of characteristic $\neq 2, 3$. Let r and s be fixed elements of A such that one of r and s is invertible in the nucleus $N(A)$ of A. Then the following properties for the (left) (r, s)-mutation $A(r, s)$ are equivalent:*
(i) $A(r, s)$ satisfies the third power identity,
(ii) $A(r, s)$ is flexible,
(iii) $A(r, s)$ is power-associative,
(iv) $A(r, s)$ is generalized quasi-alternative,
(v) $s=\alpha r$ or $r=\alpha s$ for some element α in the center $Z(A)$,
(vi) $A(r, s)\simeq A(1, \beta)$ or $A(r, s)\simeq A(\beta, 1)$ for some element β in the center $Z(A)$.

Proof. We may assume that r is invertible in $N(A)$. We have already noted the implications (iv)$\Rightarrow$(ii)$\Rightarrow$(i) and (iv)$\Rightarrow$(iii)$\Rightarrow$(i). The implication (vi)$\Rightarrow$(iv) is obvious. Assume (v) holds. Since $r\in N(A)$, the mapping $x\to xr^{-1}$ is an isomorphism of A to $A^{(r)}$. Thus we have the isomorphism $A(r, s)\simeq A^{(r)}(1, -\alpha)\simeq A(1, -\alpha)$, since $x*y=xry-\alpha yrx$. Letting $\beta=-\alpha$, we have established (vi). Therefore, it remains to show that (i) implies (v).

Assume (i) holds. Setting $y=x$ in (19), we have

$$\begin{aligned}(x, x, x)^{*} &= (xr)(xsx)+[(xrx)s]x-[(xsx)r]x-(xs)(xrx)\\ &=2(xr)(xsx)-2(xs)(xrx)=0,\end{aligned}$$

using Moufang identity (17). This gives

$$(xr)(xsx)=(xs)(xrx)$$

and replacing x by $x+1$, the terms involving x are

$$xrs+rxs+rsx=xsr+sxr+srx, \tag{20}$$

since r is in $N(A)$. The special case $x=1$ in (20) gives $sr=rs$ and hence $r^{-1}s=sr^{-1}$. Using this, (20) reduces to $sxr=rxs$ for all $x\in A$. Since r is invertible in the nucleus $N(A)$, from this we have $(r^{-1}s)x=x(sr^{-1})$ for all $x\in A$. Thus, $sr^{-1}\in K(A)$, the commutative center of A. Since $3K(A)\subseteq N(A)$ for any alternative algebra [9, p. 136] and the characteristic is not 3, we have sr^{-1} in $N(A)$ and hence $sr^{-1}\in Z(A)$. Letting $\alpha=r^{-1}s=sr^{-1}$, we have established (v). The result can be similarly proved for the case where s is invertible in $N(A)$.

REMARK. (1) In most cases of interest, the center $Z(A)$ of the algebra A consists of scalar multiples of unity 1. This is the case when A is simple over F. In this case, if r is invertible in $N(A)$ then s is invertible in $N(A)$ also. A generalized quasi-alternative algebra derived from A is quasi-alternative.

(2) Unlike the associative case, an isotope $A^{(a)}$ of an alternative algebra A is not in general isomorphic to A unless a is in $N(A)$. However, it is shown that, in a finite-dimensional simple alternative algebra A over F, any isotopes of A are isomorphic. The same result is true without the simplicity of A, if F is algebraically closed [3].

We can prove Theorem 3 under a slightly different condition that $r+s$ or $r-s$ is invertible in $N(A)$. Thus, assume that $r+s$ is invertible in $N(A)$. Let $p=(r+s)^{-1}r$. Since $r+s\in N(A)$, as for the associative case [6, p. 310], it can be shown that the mapping $f: x\rightarrow x(r+s)^{-1}$ is an isomorphism of $A(p, 1-p)$ to $A(r, s)$. Suppose $A(r, s)$ is third power-associative. Then, by Artin's theorem, the identity $(x, x, x)^*=0$ in $A(p, 1-p)$ reduces to $xpx^2-x^2px=0$. Replacing x by $1+\lambda x (\lambda\in F)$ in this, we get $0=\lambda(xp-px)+\lambda^2(x^2p-px^2)+\lambda^3(x^2px-xpx^2)$ for all $\lambda\in F$. This gives $xp=px$ for all $x\in A$ and hence p is in $K(A)$. Further, assume that A is simple over F of characteristic $\neq 2, 3$. As in the proof of Theorem 3, we have p in $Z(A)$. Since $Z(A)$ is a field, letting $\alpha=p=(r+s)^{-1}r$, we have $s=(1-\alpha)\alpha^{-1}r$ and hence r and s are invertible in A. If $r-s$ is invertible then we set $q=(r-s)^{-1}s$ and, as above, the mapping: $x\rightarrow x(r-s)^{-1}$ is an isomorphism of $A(q, q-1)$ to $A(r, s)$. Then the identity $(x, x, x)^*=0$ in $A(q, q-1)$ again implies that q is in the center $Z(A)$. Therfore, we have

THEOREM 4. *Let A be a simple alternative algebra of characteristic $\neq 2, 3$. Let r and s be fixed elements of A such that $r+s$ or $r-s$ is invertible in $N(A)$. Then r and s are invertible in A, and the properties (i)–(iv) in* Theorem 3 *and*

(v)′ $s=\alpha r$ for some invertible α in the center $Z(A)$,

(vi)′ $A(r, s)\simeq A(1, \beta)$ for some invertible β in the center $Z(A)$,

are all equivalent.

For one final remark on the equation (1) of Santilli, let A be a real or complex alternative algebra where the exponential function e^x is definable for all $x\in A$. Let r and s be fixed elements in the nucleus $N(A)$. Then the solution of (1) is given by

$$x(t)=e^{-iths}x(0)e^{itrh}. \tag{21}$$

To show that the rigt side of (21) is well defined, it suffices to verify that the subalgebra of A generated by hs, $x(0)$ and rh is associative. Since r and s are in $N(A)$, $(hsx)(rh)=[(h(sx))r]h=[h(sxr)]h=h(sxr)h=(hs)(x(rh))$, by Moufang identity. Thus the associator $(hr, x, rh)=0$ and this implies that the subalgebra of A generated by hs, x, rh is associative [1].

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