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A NOTE ON THE ALGEBRA $A_E(\Gamma)$

In Memory of Professor Dock Sang Rim

SUNGWOO SUH

1. Notations and Introduction

Let G be a locally compact abelian group with dual group Γ. We denote by $A(\Gamma)$ the algebra of all Fourier transforms of functions in $L^1(G)$. That is:

$$f\in A(\Gamma)\Leftrightarrow f(x)=\hat{F}(x)=\int_G F(y)(-y,x)\,dy$$
$$x\in\Gamma,\ y\in G,\quad F\in L^1(G)$$

with norm $\|f\|_1=\|F\|_{L^1}$. $A(\Gamma)$ is a semi-simple commutative Banach algebra and Γ is its maximal ideal space.

Let E be a compact totally disconnected subset of Γ. Define

$$A_E(\Gamma)=\{f\in A(\Gamma) : \exists U_f\supset E, \text{ open in } \Gamma, \text{ such that Card. } f(U_f)<\infty\}.$$

This algebra was used by Varopoulos in his proof that Kronecker sets are sets of strong spectral resolution.

It is clear that $A_E(\Gamma)$ is a self-adjoint subalgebra of $A(\Gamma)$. Also, for each x in Γ there is f in $A_E(\Gamma)$ such that $f(x)\neq 0$. Furthermore, $A_E(\Gamma)$ separates points on Γ.

Therefore, we may conclude by the Stone-Weierstrass theorem that $A_E(\Gamma)$ is a uniformly dense subalgebra of $C_0(\Gamma)$ as well as $A(\Gamma)$, and that $A_E(\Gamma)$ is an A-norm dense subalgebra of $A(\Gamma)$ if Γ is a totally disconnected locally compact abelian group [Th. 1.4., 5]

In the case that $\Gamma=T$ the circle group, $\boldsymbol{R}/2\pi Z$, we denote by $C^\infty(T)$ the algebra of all infinitely differentiable functions on T and define

$$C_E^\infty=\{f\in C^\infty(T) : \exists U_f\supset E, \text{ open in } T, \text{ such that Card. } f(U_f)<\infty\}.$$

It is easy to see that C_E^∞ is a separating self-adjoint subalgebra of $A(T)$, so that C_E^∞ is a uniformly dense subalgebra of $C(T)$. Also, it is well known that $C^\infty(T)$ is a dense subalgebra of $A(T)$.

J. J. Benedetto has proved the fact that if $m(E)=0$ then $\overline{C_E^\infty}=A(T)$ [2] (originally contained in [3]) using Pseudo-measures and the Hahn-Banach theorem. So we have $\overline{A_E(T)}=A(T)$ if $m(E)=0$.

Also it is a noteworthy fact which Rudin and Katznelson pointed out in their paper [3] that there exists a compact totally disconnected subset E of T with positive measure such that $\overline{C_E^\infty}=A(T)$.

2. A Sufficient Condition for $\overline{A_E(\Gamma)}=A(\Gamma)$.

Now, we will discuss this algebra in the case that Γ is not a totally disconnected locally compact abelian group. Especially, we will be concerned about the problem of finding a condition on E for $A_E(\Gamma)$ to be a dense subalgebra of $A(\Gamma)$.

For any non-discrete locally compact abelian group Γ, we find that E being a Helson and Spectral Synthesis set (which we will denote by $S-H$ set) is a sufficient condition (but not a necessary condition) for $A_E(\Gamma)$ to be a dense subalgebra of $A(\Gamma)$. If Γ is discrete, it is clear from the fact $A_E(\Gamma)$ separates points on Γ that $A_E(\Gamma)$ is a dense subalgebra of $A(\Gamma)$ for any compact totally disconnected subset E of Γ, since in this case $A_E(\Gamma)=A(\Gamma)$. Precisely, we state the theorem as follows:

THEOREM. *Suppose Γ is a non-discrete locally compact abelian group. If E is an $S-H$ set, then $A_E(\Gamma)$ is a dense subalgebra of $A(\Gamma)$.*

Proof. We will use the notation $\overline{A_E(\Gamma)}$ for the $A(\Gamma)$-norm closure of $A_E(\Gamma)$.

Since E is a Helson set, there is a constant M such that to every $\phi\in C(E)$ there corresponds an f in $A(\Gamma)$ such that $f(x)=\phi(x)$ on E and such that

(1) $$\|f\|_{A(\Gamma)}=M\|\phi\|_\infty.$$

Since E is a spectral synthesis set, that is $\overline{I_0(E)}=I(E)$, we have

(2) $$I(E)\subset\overline{A_E(\Gamma)},$$

where $I(E)=\{f\in A(\Gamma) : f\equiv 0 \text{ on } E\}$, $I_0(E)=\{f\in A(\Gamma) : f\equiv 0 \text{ on a neighborhood of } E\}$.

Consider the restriction algebra of $\overline{A_E(\Gamma)}$ to E, $\overline{A_E(\Gamma)}|_E$, which is clearly a separating self-adjoint subalgebra of $C(E)$ since E, being a Helson set, is a compact totally disconnected subset of Γ.

We shall prove that $\overline{A_E(\Gamma)}|_E$ is a uniformly closed subalgebra of $C(E)$. Once this is done, the Stone-Weierstrass theorem tells us that

$$\overline{A_E(\Gamma)}|_E=C(E)=A(\Gamma)|_E.$$

To each $f\in A(\Gamma)$ there corresponds a g in $\overline{A_E(\Gamma)}$ such that $f(x)=g(x)$ on E, so that by (2) $f-g\in\overline{A_E(\Gamma)}$. Hence, $f=g+(f-g)\in\overline{A_E(\Gamma)}$, and the theorem follows.

To prove that $\overline{A_E(\Gamma)}|_E$ is uniformly closed, let $\{g_n\}$ be a uniformly Cauchy sequence in $\overline{A_E(\Gamma)}|_E$. Passing to a subsequence (which for convenience we again write as $\{g_n\}$), we may suppose that for each n

$$\|g_{n+1}-g_n\|_\infty<2^{-n}.$$

Choose an h_n in $\overline{A_E(\Gamma)}$ such that

$$h_n(x)=g_{n+1}(x)-g_n(x) \text{ for all } x\in E, \text{ and } \|h_n\|_{A(\Gamma)}\leq M\cdot 2^{-n}.$$

this is possible because of (1) and (2). Also choose $h_0\in\overline{A_E(\Gamma)}$ such that $h_0(x)=g_1(x)$ on E.

For $n\geq 1$, let

$$f_n=\sum_{k=0}^{n-1}h_k.$$

Then, clearly, we have $f_n(x)=g_n(x)$ for all x in E. Also $\{f_n\}$ is an $A(\Gamma)$-norm Cauchy sequence in $\overline{A_E(\Gamma)}$, and so has a limit f in $\overline{A_E(\Gamma)}$. It is plain that

$$\lim_{n\to\infty}\|g_n-f|_E\|_\infty \leqq \lim_{n\to\infty}\|f_n-f\|_\infty \leqq \lim_{n\to\infty}\|f_n-f\|_{A(\Gamma)}=0.$$

Hence, $g_n \to f|_E$ uniformly on E, and this completes the proof. Q.E.D.

S. Saeki [4] has proved that every extremally disconnected compact set E in Γ (that is, the closure of any relatively open subset of E is relatively open in E) is an $S-H$ set. So, we have:

COROLLARY, *Suppose that E is an extremally disconnected compact set in Γ. Then $A_E(\Gamma)$ is a dense subalgebra of $A(\Gamma)$.* Q.E.D.

The Cantor one-third set F in T has measure 0, hence $\overline{A_F(T)}=A(T)$. And it is well known that F is a set of spectral synthesis, but not a Helson set. That is, E being an $S-H$ set is not a necessary condition for $A_E(T)$ to be a dense subalgebra of $A(T)$.

References

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Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803
U.S.A.