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HOLOMORPHIC FUNCTIONS WITH POSITIVE REAL PART ON COMPLETE CIRCULAR DOMAINS

Dedicated to the Memory of Professor Dock Sang Rim

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1. Introduction

The main purpose of the present paper is to generalize the results obtained by A. Hindmarsh in [7] to the holomorphic functions with non-negative real part defined on a complete circular domain D in certain class $\mathcal{D}$ in the complex euclidean space $\boldsymbol{C}^n$. As described in §2, $\mathcal{D}$ includes the bounded symmetric domains. More precisely, we prove the following.

THEOREM A. *A function* $f: D\to \boldsymbol{C}$, $D\in\mathcal{D}$, *is holomorphic with non-negative real part if and only if it admits an integral representation of the form*:

(1) $f(z)=i\ Im\ f(0)+\int_B[2S_D(z,\bar{b})-1]d\mu(b),\quad z\in D,$

with a positive measure μ *on the Bergman-Shilov boundary* B *such that*

(2) $\int_B g\ d\mu=0$

for all $g\in QL^2(B)^{\perp}$ *where* $S_D(z,\bar{s})$ *denotes the Szegö Kernel of* D *and the description of* $QL^2(B)$ *is given in* §2.

THEOREM B. *Let* Ω *be a non-empty open subset of* $D\in\mathcal{D}$. *If* $f:\Omega\to\boldsymbol{C}$ *is a continuous function with non-negative real part and if the function* $K:\Omega\times\Omega\to\boldsymbol{C}$, *given by*

(3) $K_\Omega(z,\bar{s})=S_D(z,\bar{s})\dfrac{f(z)+\overline{f(\bar{s})}}{2},\quad z,s\in\Omega,$

belongs to the class $P_3(\Omega)$, *see* §4 *for definition, then* f *is holomorphic in* Ω.

THEOREM C. *Let* $f:\Omega\to\boldsymbol{C}$ *be given as in* Theorem B. *If in addition* $K_\Omega\in\mathcal{P}_m(\Omega)$ *for all* $m=1, 2, \ldots,$ *and if* Re f *has a real analytic extension to* D, *then* f *admits a holomorphic extension* $F:D\to\boldsymbol{C}$ *with non-negative real part.*

Combining Theorem A and Theorem C, we obtain

THEOREM D. *Let* $M\subset D$ *be a set of uniqueness for holomorphic functions in* D, *and let* $f:M\to\boldsymbol{C}$ *be a holomorphic function which admits a real analytic extension to* D. *Then* f *admits a holomorphic extension* $F:D\to\boldsymbol{C}$ *with non-negative real part if and only if* $K_M(z,\bar{s})=S_D(z,\bar{s})\dfrac{f(z)+\overline{f(s)}}{2}$ *is positive definite on* M.

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Theorem A generalizes the well-known classical Riesz-Herglotz integral representation theorem for holomorphic functions to a complete circular domain $D\in\mathcal{D}$; this theorem is needed in the proof of Theorem D. A similar generalization has been given in [10] for polydisks.

Theorem B is a direct generalization of a remarkable result of A. Hindmarsh [7] in which he proves: If f is a continuous function in a domain D of the upper half plane in $\boldsymbol{C}^1$ with Im $f\geq 0$ and if $f\in\mathcal{P}_3(D)$ then f is holomorphic in D. It should be remarked that Theorem B still holds true under more general setting. It is easy to see that the proof of Theorem B goes through for continuous functions f defined on any open subset Ω of $\boldsymbol{C}^n$, provided that there is a continuous positive definite function $K:\Omega\times\Omega\rightarrow\boldsymbol{C}$ which is holomorphic and $K(z,\bar{z})\neq 0$ in $\Omega\times\Omega^*$, $\Omega^*=\{\bar{z}: z\in\Omega\}$. A similar result has been obtained by J. Burbea [2].

Thoerem C may be regarded as a generalization of (2) of [7], although it is considerably weaker. The existence of a real analytic extension of Re f is assumed in obtaining Theorem C. It is clear that Theorem C can be strengthed to a simply connected domain D in which there exists a positive definite function $K: D\times D\rightarrow\boldsymbol{C}$, holomorphic and $K(z,\bar{z})\neq 0$ in $D\times D^*$.

It is intereting to see if Theorems C and D can still be proved without assuming the existence of a real analytic extension of f to $D\in\mathcal{D}$. It is answered affirmatively for polydisks in [10].

2. Definitions and preliminaries

Let D be a bounded circular domain with the Bergman Shilov boundary B in the space C^n of n complex variables $z=(z_1,\ldots,z_n)$ which is complete with respect to the origin $0\in D$. D is *circular* if $z\in D$ implies $ze^{i\theta}\in D$ for $\theta\in[0,2\pi]$, and *complete* with respect to $0\in D$ if $z\in\bar{D}$ implies $rz\in D$ for $r\in[0,1)$. Assume that D admits the group G of holomorphic automorphisms. Then each $g\in G$ carries B into itself. In particular, B is invariant under the stability group $K=\{k\in G: k(0)=0\}$. Clearly, B is circular whenever D is. If K acts transitively on B, $kB=B$ for every $k\in K$ and also $gB=B$ for $g\in G$. As is well-known [3], K acts by unitary transformations. Consequently, B has a unique normalized K-invariant measure $d\sigma=V^{-1}db$, where db denotes the euclidean volume element at $b\in B$ and V the euclidean volume of B [9], [11].

By $\mathcal{D}$ we shall denote the class of all the complete bounded circular domains described above. The bounded symmetric domains form an important subclass of $\mathcal{D}$. Conversely, if any $D\in\mathcal{D}$ that admits a transitive group of holomorphic automorphisms is a bounded symmetric domain [12]. In this paper we shall consider only domains D in the class $\mathcal{D}$, unless specified otherwise.

It is well-known [8] that there exists on B a complete orthonormal system of continuous functions. Let Z_{kv} denote the monomial $z_1^{v_1}\ldots z_n^{v_n}$, $k=v_1+\ldots+v_n$, $k=0$, $1,2,\ldots,v=1,2,\ldots,m_k=\binom{n+k-1}{k}$. From the set $\{Z_{kv}\}$ we can construct a system $\Phi_0=\{\varphi_{kv}\}$, $v=1,2,\ldots,m_k$, $k=0,1,2,\ldots$, of homogeneous polynomials which is complete and orthogonal on D, and orthonormal on B. See [8]. The Szegö kernel of D is defined by the infinite series:

(1) $$S(z, \bar{s})=\sum_{k=0}^{\infty} \sum_{v=1}^{m_k} \varphi_{kv}(z)\overline{\varphi_{kv}(s)}$$

which converges uniformly on compact subsets of $D\times\bar{D}$. Therefore, $S(z,\bar{s})$ is holomorphic in $z\in D$ and antiholomorphic in $s\in D$, and continuous on $D\times\bar{D}$. Since $\varphi_{kv}(z)$ is homogeneous of order k, for each $v=1, 2, \ldots, m_k$, $\varphi_{kv}(z)=\varphi_{kv}(b)r^k$ if $z=rb$ for some $b\in B$ and $0\leq r<1$. Therefore, for $b, b'\in B$,

(2) $$S(rb', \bar{b})=\overline{S(rb, \bar{b}')}.$$

The Poisson kernel of D is defined by

(3) $$P(z, b)=\frac{|S(z, b)|^2}{S(z, z)} \quad (z\in D,\ b\in B).$$

Any holomorphic function f on D has a Fourier series expansion:

(4a) $$f(z)=\sum_{k,v} a_{kv}(f)\varphi_{kv}(z),$$

where

(4b) $$a_{kv}(f)=\lim_{r\to1} \int_B f(rb)\ \overline{\varphi_{kv}(b)}d\sigma\equiv\lim_{r\to1} (f_r, \varphi_{kv}),$$

which converges uniformly on compact subsets of D. Furthermore, we have

LEMMA 1. ([5], [6]) *Let $H^p(D)$ $(p\geq1)$ denote the usual* Hardy *space on D. If f is in the space $H^p(D)$ with the boundary value f^* on B, defined by $f^*(b)=\lim_{r\to1} f(rb)$, $b\in B$. Then f has both a Cauchy integral representation*

(5) $$f(z)=\int_B S(z, \bar{b})\ f^*(b)d\sigma\equiv(f^*, S_z)$$

and a Poisson integral representation

(6) $$f(z)=\int_B P(z, \bar{b})\ f^*(b)d\sigma\equiv(f^*, P_z)$$

for $z\in D$. Furthermore, if

$$H^p(B)=\{\tilde{f}\in L^p(B) : (\tilde{f}, S_z)=\tilde{f}, P_z)\}$$

then $H^p(B)$ is a closed subspace of $L^p(B)$ which is isometrically isomorphic to $H^p(D)$. If f^* is the boundary value of $f\in H^p(D)$, then $f^*=\tilde{f}$ a.e. on B.

It should be remarked that the system Φ_0 is not complete in the space $C(B)$ of continuous functions in general. However, according to H. Weyl [14], it is possible to extend Φ_0 to a complete orthonormal system of $C(B)$ by adding some system of functions $\Phi_1=\{\varphi_{-k} : k=1, 2, \ldots\}$. Let $\Phi=\Phi_0\cup\Phi_1$ be such a system. By letting $\varphi_k=\varphi_{k0}$ for negative $k=-1, -2, \ldots$, we can denote

(7) $\Phi=\{\varphi_{kv} : k=0, \pm1, \pm2, \ldots;\ 1\leq v\leq m_k$ for $k\geq0$; $v=0$ for $k<0\}$.

Let

$$T^2(B)=\{\tilde{f}\in L^2(B) : a_{kv}(\tilde{f})=0, \text{ for } k<0\}.$$

It is easy to see that $T^2(B)$ is a closed subspace of $L^2(B)$ which is isometrically isomorphic to $H^2(D)$ under the correspondence

(8a) $$\tau : T^2(B)\to H^2(D),$$

given by

(8b) $$\tau(\tilde{f})=\sum_{k,v} a_{kv}(\tilde{f})\ \varphi_{kv}(z)\equiv f(z).$$

If f^* is the boundary value of f, then $f^*=\tilde{f}$ a.e. on B. Namely, $T^2(B)$ may be identified with $H^2(B)$. See [6].

We also denote by $H(B)$ the class of all functions, holomorphic in D and continuous in $D\cup B$. Define the projections $P, \bar{P}$ and Q in $L^2(B)$ by

$$P : L^2(B)\to H^2(B)$$
$$\bar{P} : L^2(B)\to \bar{H}^2(B)=\{f\in L^2(B) : \bar{f}\in H^2(B)\}$$

and

$$Q : L^2(B)\to \text{Span}_C \{H^2(B), \bar{H}^2(B)\}$$

Then the space $QL^2(B)$ is the complex subspace spanned by the real parts of $H^2(B)$. If $f\in H(B)$ then clearly $f\in PL^2(B)$, $\bar{f}\in \bar{P}L^2(B)$ and $\bar{P}f(0)=f(0)$.

3. Integral Representation theorem of the Riesz-Herglotz type

In the following lemma we shall list further properties of the Szegö kernel $S(z, \bar{s})$ for later purposes.

LEMMA 2. (1^0) *$S(z, \bar{s})=\overline{S(s, \bar{z})}$ for $z, s\in D$, $S(z, \bar{z})>0$ for $z\in D$ and $S(z, 0)=1$ for $z\in D\cup B$.*

(2^0) *The function S_s defined by $S_s(z)=S(z, \bar{s})$ belongs to the class $H(B)$ for each $s\in D$ and reproduces functions $f\in H(B)$ by*

(1a) $f(z)=(f, S_z)$ $(z\in D)$.

In particular,

(1b) $\int_B S(z, \bar{b})d\sigma\equiv(1, S_z)=1$.

(3^0) *If $f\in H(B)$, then*

(2a) $\bar{f}(0)=(\bar{f}, S_z)$ $(z\in D)$

(2b) $\text{Re } f(0)=(\text{Re } f, 1)$,

and

(2c) $f(z)=i \text{ Im } f(0)+(\text{Re } f,\ 2S_z-1)$ $(z\in D)$,

(4^0) *If $f\in H(B)$, for z, $s\in D$,*

(3a) $S(z, \bar{s})f(z)=(f, S_z\bar{S}_s)$

(3b) $(\text{Re } f, S_z\bar{S}_s)=S(z, \bar{s})\,\dfrac{f(z)+\overline{f(s)}}{2}$

and

(3c) $(\text{Im } f,\ S_z\bar{S}_s)=S(z, \bar{s})\,\dfrac{f(z)-\overline{f(s)}}{2i}$.

(5^0) *For $z, s\in D$,*

(4a) $P(S_z\bar{S}_s)=S(s, \bar{z})S_z$

(4b) $\bar{P}(S_z\bar{S}_s)=S(s, \bar{z})\bar{S}_s$

and

(4c) $Q(S_z\bar{S}_s)=S(s, \bar{z})(S_z+\bar{S}_s-1)$.

Proof. Properties (1^0) and (2^0) are evident.
(3^0) Since P is a projection of $L^2(B)$ onto $H^2(B)$, $PS_z=S_z$ and $P^2=P$. This together with (1b) implies (2a):

$$(\bar{f}, S_z)=(\bar{f}, PS_z)=(P\bar{f}, S_z)=(\overline{f(0)}, S_z)=\overline{f(0)}(1, S_z)=\overline{f(0)}.$$

Equality (2b) follows by adding

$$f(0)=(f, S_0)=(f, 1)$$

and its complex conjugate.

Adding (1a) and (2a), we have

$$f(z)=-\mathrm{Re} f(0)+i\,\mathrm{Im} f(0)+(2\mathrm{Re}\ f, S_z).$$

This together with (2b) implies (2c).

(4^0) For each $s\in D$, $S_s\,f\in H(B)$. Applying (1a) to $S_s f$, we have

$$S(z, \bar{s})f(z)=(S_s f, S_z)=(f, S_z\bar{S}_s).$$

Taking the complex conjugate to the expression obtained from (3a) by exchanging the roles of z and s we also obtain

$$S(z, \bar{s})\overline{f(s)}=(\bar{f}, S_z\bar{S}_s).$$

Adding and subtracting these two relations, we have both relations (3b) and (3c), respectively.

(5^0) If $f\in H(B)$, by (1a),

$$(f, S(s, \bar{z})S_z)=\overline{S(s, \bar{z})}(f, S_z)=S(z, \bar{s})f(z).$$

This relation and (3a) together imply

$$(f, S_z\bar{S}_s)=(f, S(s, \bar{z})S_z)$$

for all $f\in H(B)$. Since $S(s, \bar{z})S_z\in H(B)$, (4a) follows. A similar proof can be given for (4b). Relations (3b) and (2c) yield

$$(\mathrm{Re}\ f, S_z\bar{S}_s)=(\mathrm{Re}\ f, S(s, \bar{z})(S_z+\bar{S}_s-1)).$$

Since $\{\mathrm{Re}\ f: f\in H^2(B)\}$ spans $QL^2(B)$ and $S_z+\bar{S}_z-1\in QL^2(B)$, the assertion (4c) follows.

Lemma 2 leads to the proof of Theorem A.

Proof of Theorem A. Suppose that $f: D\to C$ is a holomorphic function with non-negative real part in D. Then for $r\in(0, 1)$ the function $f_r, f_r(z)=f(rz)$, is continuous and holomorphic in $\bar{D}$. By (3°) of Lemma 2,

$$f_r(z)=i\,\mathrm{Im}\, f_r(0)+\int_B[2S(z, \bar{b})-1]d\mu_r(b),$$

where

$$d\mu_r(b)=\mathrm{Re}\, f_r(b)d\sigma$$

is a positive measure on B. Clearly, we have

$$\int_B g(b)d\mu_r(b)=0$$

for all $g\in(QL^2)^\perp(B)$ and the total variation of μ_r is bounded as $r\to1$. In fact,

(5) $\int_B d\mu_r(b)=\int_B[2S(0, b)-1]\ \mathrm{Re}\ f_r(b)d\sigma$
$=\mathrm{Re}\ f_r(0)=\mathrm{Re}\ f(0).$

By Helly's selection theorem, $\{\mu_r(b)\}$ has a subsequence which converges everywhere on B to $\mu(b)$ of bounded variations such that

$$\lim_{r\to1} f_r(z)=i\ \mathrm{Im}\ f(0)+\int_B[2S(z, \bar{b})-1]d\mu(b),$$

as desired.

Conversely, if f is defined as in (1) of §1, then it is holomorphic in D, since $S(z,\bar{s})$ is holomorphic in $z\in D$ and continuous in $\bar{D}$. Therefore, it remains to show that Re $f\geq 0$. If $f\in L^2(B)$ has an absolutely and uniformly convergent series expansion in terms of the complete system Φ.

Then by (2) of §1,

(6) $\int_B (Qf)(b)d\mu(b)=\int_B f(b)d\mu(b)$,

and (6) is satisfied by the function $f=S_z\bar{S}_z$, that is,

(7) $\int_B Q(S_z\bar{S}_z)(b)d\mu(b)=\int_B(S_z\bar{S}_z)(b)d\mu(b)$.

Taking the real part of (1), §1, we have

$$\begin{aligned}\text{Re } f(z)&=\int_B[S(z,\bar{b})+\overline{S(z,\bar{b})}-1]d\mu(b)\\&=S(z,\bar{z})^{-1}\int_B Q(S_z\bar{S}_z)(b)d\mu(b), \text{ by (4c)}.\\&=S(z,\bar{z})^{-1}\int_B(S_z\bar{S}_z)(b)d\mu(b), \text{ by (7)}.\\&=S(z,\bar{z})^{-1}\int_B|S(z,\bar{b})|^2d\mu(b)>0,\end{aligned}$$

completing the proof of Theorem A.

4. Holomorphic extensions and positive definite functions

Let S be any topological space. By $\mathcal{P}_m(S)$ we shall denote the class of all continuous hermitian symmetric functions:

$$K: S\times S\rightarrow C$$

which satsfies the relation:

$$(1)\qquad \sum_{i,j=1}^{m} K(x_i, x_j)\alpha_i\bar{\alpha}_j\geq 0$$

for any choice of m points $x_1, \ldots, x_m\in S$ and complex numbers $\alpha_1, \ldots, \alpha_m$. A function K in $\mathcal{P}_m(S)$ is called a *positive definite function of order m*. A *positive definite function* on S is then defined as a positive definite function of all orders $m=1, 2, \ldots$. The class of all positive definite functions on S is denoted by $\mathcal{P}(S)$.

It is well-known [1] that any positive definite function $K\in\mathcal{P}(S)$ determines a Hilbert space $H(S)$ uniquely and enjoys the following properties:

(a) $K(x, y)=Ky(x)$ reproduces all $f\in H(S)$, i.e.,

$$f(x)=\langle f, K_x\rangle \quad (x\in S),$$

where $\langle\ ,\ \rangle$ denotes the inner product in $H(S)$.

(b) There exists a complete orthonormal system $\{\varphi_v\}_{v=1}^{\infty}$ such that

(2) $K(x, y)=\sum_{v=1}^{\infty}\varphi_v(x)\overline{\varphi_v(y)}$.

Conversely, it is easy to see that if a sequence of functions $\{\varphi_v\}$ is given on S with the propety:

(3) $\sum_{v=1}^{\infty}|\varphi_v(x)|^2<\infty \quad (x\in S)$,

then the function

(4) $K(x, y)=\sum_{v=1}^{\infty}\varphi_v(x)\overline{\varphi_v(y)}$

belongs to $\mathcal{P}(S)$.

A positive definite function $K\in\mathcal{P}(S)$ which reproduces a Hilbert space $H(S)$ is called *the kernel function* for the Hilbert space. In particular, if S is replaced by a complete circular domain $D\in\mathfrak{D}$ with Bergman-Shilov boundary B and the kernel K by the Szegö kernel $S(z,\bar{s})$, then its associated Hilbert space is the Hardy space $H^2(D)$.

To prove Theorem B we need the following preparatory lemma which is a complex version of the main lemma in [7]. For completeness sake we give a proof of this lemma here. See also [2].

LEMMA 3. *Let Ω be an open set in $\boldsymbol{C}^n$ and let $K:\Omega\times\Omega\to\boldsymbol{C}$ be a C^2 function which belongs to the class $\mathcal{P}_{2m+1}(\Omega)$ $(1\leq m\leq n)$. Then the $(2m+1)\times(2m+1)$ matrix*

$$(5)\quad \tilde{M}_{2m+1}(u,v)\equiv\begin{pmatrix} K & \bar{\partial}_{v^1}K & \partial_{v^1}k \;\cdots\cdots\; \bar{\partial}_{v^m}K & \partial_{v^m}K \\ \partial_{u^1}K & \partial_{u^1}\bar{\partial}_{v^1}K & \partial_{u^1}\partial_{v^1}K \cdots\cdots \partial_{u^1}\bar{\partial}_{v^m}K & \partial_{u^1}\partial_{v^m}K \\ \bar{\partial}_{u^1}K & \bar{\partial}_{u^1}\bar{\partial}_{v^1}K & \bar{\partial}_{u^1}\partial_{v^1}K \;\cdots\cdots\cdots & \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{u^m}K & \cdot & \cdot & \cdot \\ \bar{\partial}_{u^m}K & \bar{\partial}_{u^m}\bar{\partial}_{v^1}K \cdots\cdots\cdots & \bar{\partial}_{u^m}\bar{\partial}_{v^m}K & \bar{\partial}_{u^m}\partial_{v^m}K \end{pmatrix}.$$

is positive definite at every $(z,s)\in\Omega\times\Omega$, where $\partial_u{}^k=\sum_{j=1}^n u_j{}^k\frac{\partial}{\partial z_j}$, $\bar{\partial}_u{}^k=\sum u_j{}^k\frac{\partial}{\partial\bar{z}_j}$ and $u^k, v^k (k=1,2,\ldots\ldots,m, 1\leq m\leq n)$ are vectors in $\boldsymbol{R}^n$.

Proof. Let $(z,s)\in\Omega\times\Omega$ be a fixed point with $z=x+i\xi$, $s=y+i\eta$ for some x,ξ,y,η in $\boldsymbol{R}^n$. Applying the main lemma of [7] to K at $(z,s)=((x,\xi),(y,\eta))\in\Omega\times\Omega$, we can construct the following $(2m+1)\times(2m+1)$ positive definite matrix

$$(6)\quad M_{2m+1}=\begin{pmatrix} K & \nabla_{v^1}K & \nabla_{\nu^1}K \cdots\cdots \nabla_{v^m}K & \nabla_{\nu^m}K \\ \nabla_{u^1}K & \nabla_{u^1}\nabla_{v^1}K & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \nabla_{u^1}\nabla_{\nu^m}K \\ \nabla_{\mu^1}K & & \\ \vdots & & \vdots \\ \nabla_{u^m}K & & \cdot \\ \nabla_{\mu^m}K & \nabla_{u^m}\nabla_{v^1}K \cdots\cdots\cdots\cdots\cdots\cdots & \nabla_{\mu^m}\nabla_{\nu^m}K. \end{pmatrix}$$

where $u^k=(u_1{}^k,\ldots,u_n{}^k,0\ldots0)$
$\mu^k=(0,\ldots,0,u_1{}^k,\ldots,u_n{}^k)$
$v^k=(v_1{}^k,\ldots,v_n{}^k,0,\ldots0)$
$\nu^k=(0,\ldots,0,v_1{}^k,\ldots,v_n{}^k)$

and

$$(7)\quad \begin{pmatrix} \nabla_{u^k}K=\sum_{j=1}^n u_j{}^k\dfrac{\partial K}{\partial x_j} \\ \nabla_{\mu^k}K=\sum_{j=1}^n u_j{}^k\dfrac{\partial K}{\partial\xi_j} \\ \nabla_{v^k}K=\sum_{j=1}^n v_j{}^k\dfrac{\partial K}{\partial y_j} \\ \nabla_{\nu^k}K=\sum_{j=1}^n v_j{}^k\dfrac{\partial K}{\partial\eta_j} \end{pmatrix}$$

are evaluated at $((x,\xi),\ (y,\eta))$.

Let B_{2m+1} be an invertible $(2m+1)\times(2m+1)$ matrix of the form:

$$(8)\quad B=B_{2m+1}=\begin{pmatrix} 1 & 0\cdots\cdots\cdots\cdots 0 & \\ 0 & J_2 & 0 \\ \vdots & & \vdots \\ 0\cdots\cdots\cdots\cdots & 0 & J_2 \end{pmatrix}$$

where $J_2=\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$. A straight forward calculation shows that $BM_{2m+1}B^*=\tilde{M}_{2m+1}(u, v)$ and hence $\tilde{M}_{2m+1}$ is positive definite.

Using Lemma 3, we can prove Theorem B.

Proof of Theorem B. We consider the case where $f: \Omega\to \boldsymbol{C}$ is in C^2 and apply Lemma 3 to the function K_Ω defined by (3), § 1, with $m=1$ and $u^j=v^j=e^j=(\delta_i^j)_{1\leq i\leq n}$, $j=1, 2, \ldots n$, where $\delta_j^i=1$ for $i=j$, $=0$ for $i\neq j$. Then

$$(9)\quad \tilde{M}_3(e^j, e^j)=\begin{pmatrix} K & \dfrac{\partial K}{\partial \bar{s}_j} & \dfrac{\partial K}{\partial s_j} \\ \dfrac{\partial K}{\partial z_j} & \dfrac{\partial^2 K}{\partial z_j \partial \bar{s}_j} & \dfrac{\partial^2 K}{\partial z_j \partial s_j} \\ \dfrac{\partial K}{\partial \bar{z}_j} & \dfrac{\partial^2 K}{\partial \bar{z}_j \partial \bar{s}_j} & \dfrac{\partial^2 K}{\partial \bar{z}_j \partial s_j} \end{pmatrix}$$

is positive definite at every point $(z, s)\in\Omega\times\Omega$ for all $j=1, 2, \ldots, n$. A simple computation leads to

$$(10)\quad \begin{aligned} &\frac{\partial K}{\partial \bar{z}_j}=\frac{1}{2}S_D(z, \bar{s})\frac{\partial f}{\partial \bar{z}_j} \\ &\frac{\partial K}{\partial s_j}=\frac{1}{2}S_D(z, \bar{s})\frac{\partial \bar{f}}{\partial s_j} \\ &\frac{\partial^2 K}{\partial \bar{s}_j \partial z_j}=\frac{1}{2}\ \frac{\partial S_D(z, \bar{s})}{\partial z_j}\ \frac{\partial \bar{f}}{\partial s_j} \\ &\frac{\partial^2 K}{\partial \bar{z}_j \partial s_j}=0. \end{aligned}$$

Since the matrix $\tilde{M}_3(e^j, e^j)$ is positive definite, we have

$$(11)\quad K\frac{\partial^2 K}{\partial \bar{z}_j \partial s_j}-\frac{\partial K}{\partial \bar{z}_j}\ \frac{\partial K}{\partial s_j}\geq 0.$$

Evaluating (11) at $(z, z)\in\Omega\times\Omega$, we have

$$(12)\quad \frac{1}{4}S_D{}^2(z, \bar{z})\left|\frac{\partial f}{\partial \bar{z}_j}\right|^2\leq 0,$$

which implies $\dfrac{\partial f}{\partial \bar{z}_j}=0$ for all $j=1, 2, \ldots, n$, since $S_D(z, \bar{z})>0$ on $\Omega\times\Omega$. Therefore, f is holomrophic in Ω. To complete the proof of the theorem, we need to consider the case where f is merely continuous on Ω. It can be done as follows. If f is locally integrable on Ω then for each $\epsilon>0$ there exists a smooth function $f_\epsilon\in C^\infty(\Omega)$. If in addition f is continuous in Ω then $\lim_{\epsilon\to 0} f_\epsilon=f$ uniformiy on compact subsets of Ω. Since the property of a function being in $\mathcal{P}_3(\Omega)$ is additive and positively homogeneous, both $K_D(z, \bar{s})$ and $K_D^{(\epsilon)}(z, \bar{s})=S_D(z, \bar{s})\dfrac{f_\epsilon(z)+\overline{f_\epsilon(s)}}{2}$ belong to $\mathcal{P}_3(\Omega)$. Therefore, $f_\epsilon\in C^\infty(\Omega)$ is holomorphic in Ω by the previous result. By the uniform convergence, f, too, is holomorphic on Ω. This completes the proof.

We shall call a set $M\subset D$ *a set of uniqueness* (for holomorphic functions on D) whenever a function f holomorphic in D vanishes on M vanishes everywhere in D.

For example, any non-empty open subset of a simply connected domain G in $\boldsymbol{C}^n$ is a set of uniqueness for holomorphic functions in G for $n>1$, while in $\boldsymbol{C}$ a set with an accumulation point is enough to be a set of uniqueness for holomorphic functions defined in a simply connected domain containing the set and an accumulation point.

Theorem C is contained in the following slightly more general theorem.

THEOREM C$'$. *Let M be a set of uniqueness of D in which the function $f: M\to\boldsymbol{C}$ is holomorphic with values in* Re $f\geq 0$. *If* Re f *has a real analytic extension to D and if* $K_M(z,\bar{s})=S_D(z,\bar{s})\dfrac{f(z)+\overline{f(s)}}{2}$, z, $s\in M$, *is positive definite in M,* i.e., $K_M\in\mathcal{P}(M)$, *then f admits a holomorphic extension $F: D\to\boldsymbol{C}$ with values in* Re $F\geq 0$.

Furthermore, let $H(M)$ and $H(D)$ be the Hilbert spaces associated with K_M and $K_D(z,\bar{s})=S_D(z,\bar{s})\dfrac{F(z)+\overline{F(s)}}{2}$. *Then there is a natural isometry between these Hilbert spaces.*

Proof. Since $K_M(z,\bar{s})$ is positive definite and holomorphic in $(z,\bar{s})\in M\times M^*$, $M^*=\{\bar{z}: z\in M\}$, there exists a Hilbert space $H(M)$ of holomorphic functions on M. Let $\tilde{H}(M)$ be the subspace of $H(M)$ consisting of all finite linear combinations of the form:

(13) $u(z)=\sum_{J\in J} a_j K_M(z,\bar{s}_j)$ for $s_j\in M$,

where J denotes a finite index set. Since Re $f(z)$ has a real analytic extension to D, so does the function $K_M(z,\bar{z})=S_D(z,\bar{z})$ Re $f(z)$. By a result of [13], $K_M(z,\bar{s})$ has a unique holomorphic extension to $K_D(z,\bar{s})$, $(z,\bar{s})\in D\times D^*$, and $K_D\in\mathcal{P}(D)$. Let $H(D)$ be the Hilbert space associated with K_D and $\tilde{H}(D)$ the subspace of $H(D)$ determined by exactly the same linear combinations as for the space $\tilde{H}(M)$. Then

$$\begin{aligned}(14)\qquad \|u\|_D{}^2&=\langle u,u\rangle_D=\Big\langle\sum_{j\in J} a_j M_M(z,s_j),\ \sum_{k\in J} a_k K_M(z,s_k)\Big\rangle\\ &=\sum_{j,k\in J} a_j\bar{a}_k K_M(s_j,\bar{s}_k)=\|u\|_M{}^2.\end{aligned}$$

Therefore, a function in $\tilde{H}(M)$ admits a holomorphic extension to a function in $\tilde{H}(D)$ having the same norm. On the other hand, the collection of functions $K_s=K_D(\cdot,\bar{s})$, $s\in M$, spans a linear subspace of $H(D)$ which is dense in that space, In fact, if $f\in H(M)$ is orthogonal to all such K_s, $s\in M$, then $f(s)=\langle f, K_s\rangle=0$ for all $s\in M$.

Since M is a set of uniqueness, $f(s)\equiv 0$ in D. Therefore, $\tilde{H}(M)$ is dense in $H(D)$ as well as in $H(M)$. Thus, there is a natural isometry between $H(M)$ and $H(D)$. Now it remains to show that Re $F\geq 0$ for all $F\in H(D)$. But it is immediate from the fact that Re $F(z)=K_D((z,\bar{z})\cdot S_D(z,\bar{z})^{-1}$ and $K_D(z,\bar{z})\geq 0$ for all $z\in D$.

Proof of Theorem D. The sufficiency has already been proven in Theorem C$'$. The necessity follows easily from the integral representation in Theorem A. Suppose that

$F: D \to C$ is holomorphic with Re $F \geq 0$. Consider the function $K_r: D \times D \to C$ defined by

$$K_r(z, \bar{s}) = S_D(z, \bar{s}) \frac{F_r(z) + \overline{F_r(s)}}{2}$$

for $r \in (0, 1)$. For any positive integer n, let $z^1, \ldots z^n \in D$ and $\alpha_1, \ldots, \alpha_n \in C$. Then

$$(15) \quad \sum_{j,k=1}^{n} K_r(z^j, z^k) \alpha_j \bar{\alpha}_k = \sum_{j,k=1}^{n} (\text{Re } F_r, \ S_{z^j} \bar{S}_{z^k}) \alpha_j \bar{\alpha}_k = \int_B \Big| \sum_{j=1}^{n} \alpha_j S(z^j, b) \Big|^2 \text{Re } F_r(b) d\sigma \geq 0.$$

Since (15) holds for all $r \in (0, 1)$ and $K_r(z, \bar{s})$ is a continuous function of r for all fixed z and s in D, it follows that

$$\sum_{j,k=1}^{n} K(z^j, z^k) \alpha_j \bar{\alpha}_k \geq 0$$

for all integers n, i.e., K is positive definite in D and, hence in M.

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