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TABLES OF D-CLASSES IN THE SEMIGROUP B_n OF THE BINARY RELATIONS ON A SET X WITH n-ELEMENTS

In Memory of Professor Dock Sang Rim

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0. Introduction

$M_n(F)$ denotes the set of all $n\times n$ matrices over $F=\{0,1\}$. For $a,b\in F$, define $a+b=\max\{a,b\}$ and $ab=\min\{a,b\}$. Under these operations $a+b$ and ab, $M_n(F)$ forms a multiplicative semigroup (see [1], [4]) and we call it the semigroup of the $n\times n$ Boolean matrices over $F=\{0,1\}$. Since the semigroup $M_n(F)$ is the matrix representation of the semigroup B_n of the binary relations on the set X with n elements, we may identify $M_n(F)$ with B_n for finding all D-classes. A problem [3] is for the determination of all D-classes in B_n, we pose the following problem.

PROBLEM 1. Find all D-classes in $M_n(F)$.

This is an unsolved problem for $n>4$. We shall give tables of all D-classes in M_n (F) for $n=1,2,3,4$.

As the second item of this paper we pose problems in connection with the semigroup $M_n(F)$ of all $n\times n$ Boolean matrices over F.

1. Tables of D-classes

Let $F=\{0,1\}$ be the set of two elements 0 and 1. $M_n(F)$ denotes the semigroup of all $n\times n$ Boolean matrices over F.

DEFINITION 1. Define a set $V_n(F)=\{(x_0x_1\cdots x_{n-1}) : x_i\in F\}$. For $(x_0x_1\cdots x_{n-1})\in V_n(F)$ we define $m(x_0x_1\cdots x_{n-1})=m_i=\sum_{i=0}^{n-1} x_i2^i$, (a positive integer m_i) and we shall say that the name of $(x_0x_1\cdots x_{n-1})$ is m_i.

EXAMPLE 1. Let $(101)\in V_3(F)$. We call (101) 5 and write $m(101)=5=1\cdot 2^0+0\cdot 2+1\cdot 2^2$. We have that $11=m(1101)$. For $A=(a_{ij})\in M_n(F)$, we denote by A_i the i-th row of A. Since $A_i\in V_n(F)$, there exists m_i such that $m(A_i)=m_i$.

DEFINITION 2. Let $A\in M_n(F)$ and let A_i be the i-th row of A. Let $m_i=m(A_i)$ be the name of A_i. We call $A(m_1m_2\cdots m_n)$ and write $A=(m_1m_2\cdots m_n)$.

EXAMPLE 2. Let $A=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}\in M_3(F)$. Since $m(A_1)=m(1\,1\,0)=3$, $m(A_2)=m(1\,0\,1)$

$=5$, $m(A_3)=m(0\,1\,1)=6$, we can write $A=(3\,5\,6)$.

NOTATION. We shall have the following notation:

11(3 5 6)
$r=3$, $c=3$
NR
$n=6$
$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$$

The above notation means that the D-class in $M_3(F)$ containing $A=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$ has 6 ($n=6$) elements, non-regular (NR) and 11th D-class in a table for $M_3(F)$. As we noted above the name of A is $(3\,5\,6)$, $r=3$ means that the row rank of A is equal to 3, and $c=3$ means that the column rank of A is equal to 3. We have now D-classes tables in $M_n(F)$, $n=1,2,3,4$.

TABLE 1. $M_1(F)$ has two D-classes:

1(0) (0)	2(1) (1)
$r=0$, $c=0$	$r=1$, $c=1$
R	R
$n=1$	$n=1$

R denotes that *it is regular*.

TABLE 2. $M_2(F)$ has 4 D-classes.

1(0 0)	2(1 0)	3(1 2)	4(1 3)
$r=0=c$	$r=1=c$	$r=2=c$	$r=2=c$
R	R	R	R
$n=1$	$n=9$	$n=2$	$n=4$
$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$

TABLE 3. $M_3(F)$ has 11 D-classes.

1(0 0 0), $r=0=c$, R, $n=1$, $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	2(1 0 0), $r=1=c$, R, $n=49$, $\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$	3(1 2 0), $r=1=c$, R, $n=162$, $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$
4(1 3 0), $r=2=c$, R, $n=144$, $\begin{pmatrix}1&0&0\\1&1&0\\0&0&0\end{pmatrix}$	5(1 2 4), $r=3=c$, R, $n=6$, $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	6(1 2 5), $r=3=c$, R, $n=36$, $\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}$
7(1 2 7), $r=3=c$, R, $n=18$, $\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}$	8(1 3 5), $r=3=c$, R, $n=18$, $\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}$	9(1 3 7), $r=3=c$, R, $n=36$, $\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$
10(1 3 6), $r=3=c$, NR, $n=36$, $\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}$	11(3 5 6), $r=3=c$, NR, $n=6$, $\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$	

TABLE 4. $M_4(F)$ has 60 D-classes.

1(0000), $r=0=c$, R, $n=1$, $\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	2(1000), $r=1=c$, R, $n=225$, $\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	3(1200), $r=2=c$, R, $n=6050$, $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

4(1300) $r=2=c$ R $n=2500$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

5(1240) $r=3=c$ R $n=4056$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$

6(1250) $r=3=c$ R $n=11664$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}$

7(1270) $r=3=c$ R $n=3528$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\0&0&0&0\end{pmatrix}$

8(1350) $r=3=c$ R $n=3528$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}$

9(1360) $r=3=c$ NR $n=7056$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$

10(1370) $r=3=c$ R $n=3600$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&0&0&0\end{pmatrix}$

11(3560) $r=3=c$ NR $n=1176$ $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$

12(35 10 0) $r=3,\ c=4$ NR $n=1584$ $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&0&0\end{pmatrix}$

13(35140) $r=3,\ c=4$ NR $n=1296$ $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&1\\0&0&0&0\end{pmatrix}$

14(1248) $r=4=c$ R $n=24$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

15(1249) $r=4=c$ R $n=288$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}$

16(12411) $r=4=c$ R $n=288$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&1&0&1\end{pmatrix}$

17(12415) $r=4=c$ R $n=96$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&1&1&1\end{pmatrix}$

18(1256) $r=4,\ c=3$ NR $n=1584$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&1&1&0\end{pmatrix}$

19(1259) $r=4=c$ R $n=288$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}$

20(12510) $r=4=c$ R $n=288$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix}$

21(12511) $r=4=c$ R $n=576$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&1&0&1\end{pmatrix}$

22(12512) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&1&1\end{pmatrix}$

23(12513) $r=4=c$ R $n=576$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&0&1&1\end{pmatrix}$

24(12514) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&1&1&1\end{pmatrix}$

25(12515) $r=4=c$ R $n=576$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&1&1&1\end{pmatrix}$

26(12711) $r=4=c$ R $n=144$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&0&1\end{pmatrix}$

27(12712) $r=4=c$ NR $n=288$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\0&0&1&1\end{pmatrix}$

28(12713) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&0&1&1\end{pmatrix}$

29(12715) $r=4=c$ R $n=288$ $\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}$

30(1256) $r=4,\ c=3$ NR $n=1296$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\0&1&1&0\end{pmatrix}$

31(1359) $r=4=c$ R $n=96$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}$

32(13510) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix}$

33(13511) $r=4=c$ R $n=576$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&0&1\end{pmatrix}$

34(13514) $r=4=c$ NR $n=288$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\0&1&1&1\end{pmatrix}$

35(13515) $r=4=c$ R $n=288$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&1&1\end{pmatrix}$

36(13610) $r=4=c$ NR $n=288$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&1&0&1\end{pmatrix}$

37(13611) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\1&1&0&1\end{pmatrix}$

38(13612) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}$

39(13613) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\1&0&1&1\end{pmatrix}$

40(13614) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&1&1&1\end{pmatrix}$

41(13615) $r=4=c$ NR $n=576$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\1&1&1&1\end{pmatrix}$

42(13711) $r=4=c$ R $n=288$ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&0&1\end{pmatrix}$

43(13712), $r=4=c$, NR, $n=576$: $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&0&1&1\end{pmatrix}$

44(13713), $r=4=c$, NR, $n=576$: $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&0&1&1\end{pmatrix}$

45(13714), $r=4=c$, NR, $n=576$: $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&1&1&1\end{pmatrix}$

46(13715), $r=4=c$, R, $n=576$: $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}$

47(161012), $r=4=c$, NR, $n=96$: $\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&1&0&1\\0&0&1&1\end{pmatrix}$

48(161013), $r=4=c$, NR, $n=288$: $\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&1&0&1\\1&0&1&1\end{pmatrix}$

49(161113), $r=4=c$, NR, $n=288$: $\begin{pmatrix}1&0&0&0\\0&1&1&0\\1&1&0&1\\1&0&1&1\end{pmatrix}$

50(171113), $r=4=c$, NR, $n=96$: $\begin{pmatrix}1&0&0&0\\1&1&1&0\\1&1&0&1\\1&0&1&1\end{pmatrix}$

51(3569), $r=4=c$, NR, $n=288$: $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}$

52(35611), $r=4=c$, NR, $n=72$: $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\1&1&0&1\end{pmatrix}$

53(35615), $r=4=c$, NR, $n=96$: $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\1&1&1&1\end{pmatrix}$

54(35914), $r=4=c$, NR, $n=96$: $\begin{pmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\0&1&1&1\end{pmatrix}$

55(351012), $r=4=c$, NR, $n=72$: $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&1\end{pmatrix}$

56(351013), $r=4=c$, NR, $n=576$: $\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\1&0&1&1\end{pmatrix}$

57(351114), $r=4=c$, NR, $n=576$: $\begin{pmatrix}1&1&0&0\\1&0&1&0\\1&1&0&1\\0&1&1&1\end{pmatrix}$

58(371213), $r=4=c$, NR, $n=288$: $\begin{pmatrix}1&1&0&0\\1&1&1&0\\0&0&1&1\\1&0&1&1\end{pmatrix}$

59(371314), $r=4=c$, NR, $n=288$: $\begin{pmatrix}1&1&0&0\\1&1&1&0\\1&0&1&1\\0&1&1&1\end{pmatrix}$

60(711 13 14), $r=4=c$, NR, $n=24$: $\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}$

2. Fuzzy matrix semigroups and boolean matrix semigroups

Kim [4] initiated a new class of semigroups $M_n(K)$, (the semigroups of the $n\times n$ fuzzy matrices over $K=\{r_i\in[0,1] : i=0,1,2,\dots,m\}$) and Kim [7] studied a class of semigroups $M_n(S)$ of the $n\times n$ Boolean matrices over the set $S=[a,b)$ of the interval, where a and b are two real numbers such that $a<b$. The number of all D-classes in the semigroup $M_2(K)$ is given by Theorem [5]. Therefore we pose a problem.

PROBLEM 2. Find all D-classes of $M_n(K)$.

Let $M_n(S)$ be the semigroup of the $n\times n$ Boolean matrices over a set $S=[a,b)$. Let $A\in M_n(S)$ and let I be the the identity matrix in $M_n(S)$. If there exists B in $M_n(S)$ such that $AB=BA=I$, then we say that B is the inverse of A and A is invertible. Kim [7] established a characterization of an invertible Boolean matrix in $M_n(S)$.

PROBLEM 3. Give a characterization of all idempotent Boolean matrices in $M_n(S)$.

In [9] we have a semigroup $H(X)$ of all choice functions on a finite set X. In connection with $H(X)$ [9], we pose the following problem.

PROBLEM 4. Find the total number of the idempotent choice functions in $H(X)$. (See Theorem 3[9]).

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