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ON THE FRUM-KETKOV TYPE FIXED POINT THEOREMS

Dedicated to the memory of Professor Dock Sang Rim

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In [3], Frum-Ketkov obtained a fixed point theorem, which was later generalized by Nussbaum [6] and Buley [2]. In this paper, we obtain some other forms of the Frum-Ketkov theorem and a variation of a theorem of Belluce and Kirk [1].

In [6], Nussbaum obtained generalizations of the following fixed point theorem:

THEOREM A (Frum-Ketkov [3, Theorem 3]). *Let G be a closed convex subset of a Banach space X and $f : G \to G$ be a continuous map. Assume there exists a nonempty compact subset M of X and*

(a) *there is a constant $c<1$ such that for all $x \in G$,*

$$d(fx, M) \leq c\, d(x, M).$$

Then f has a fixed point.

We first obtain a number of extensions of Theorem A by weakening the condition (a).

Motivated by that the condition (a) resembles the Banach contraction, we consider the following conditions:

(b) For any $\varepsilon>0$ there exists a $\delta>0$ and an ε_0 with $0<\varepsilon_0<\varepsilon$ such that for any $x \in G$, $0 \leq d(x,M) < \varepsilon+\delta$ implies $d(fx,M) \leq \varepsilon_0$.

(b)′ There exists a nondecreasing map $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ which is continuous from the right such that $\phi(t)<t$ for all $t>0$ and $d(fx,M) \leq \phi(d(x,M))$ for any $x \in G$.

(c) For any $\varepsilon>0$ there exists a $\delta>0$ and an ε_0 with $0<\varepsilon_0<\varepsilon$ such that for any $x \in G$, $\varepsilon \leq d(x,M) < \varepsilon+\delta$ implies $d(fx,M) \leq \varepsilon_0$.

(c)′ There exists a map $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ which is upper semicontinuous from the right on $\mathbf{R}_+ - \{0\}$ such that $\phi(t)<t$ for all $t>0$ and for any $x \in G$, $d(fx,M) \leq \phi(d(x,M))$.

(d) For any $\varepsilon>0$ there exists a $\delta>0$ such that for any $x \in G$, $\varepsilon \leq d(x,M) < \varepsilon+\delta$ implies $d(fx,M) < \varepsilon$.

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Note that (b) $\Longleftrightarrow$ (b)$'$, (c) $\Longleftrightarrow$ (c)$'$, (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d), and these implications are proper. (See Hegedüs and Szilágyi [4] and Park [7]). Note also that the condition (d) is a modification of a contractive condition given by Meir and Keeler [5].

Now we show that Theorem A remains true if we replace the condition (a) by one of (b), (c), and (d). Consequently, we obtain proper extensions of Theorem A.

Our tool is the following generalization of Theorem A.

THEOREM B. (Nussbaum [6, Corollary 1]) *Let G be a closed convex subset of a Banach space X and $f: G \to G$ a continuous map. Assume that there exists a compact set $M \subset X$ and*

(1) for each open neighborhood G_0 of M in X and each $x \in G$, there exists an integer $n_0(x_0, G_0)$ such that $f^n(x) \in G_0$ for $n \geq n_0$, and

(2) there exist two sequences of positive numbers $\{a_k\}$ and $\{b_k\}$ with $a_k > b_k$ and $a_k \to 0$ such that f maps $N_{a_k}(M) \equiv \{x \in G : d(x, M) \leq a_k\}$ into $N_{b_k}(M)$ for all $k \geq 1$.

Then f has a fixed point.

The following is our first result.

THEOREM 1. *Let G be a closed convex subset of a Banach space X and f a continuous selfmap of G. If there exists a nonempty compact set $M \subset X$ satisfying $d(fx, M) \leq d(x, M)$ for $x \in G$ and the condition* (d), *then f has a fixed point.*

Proof. Note that for any $x \in G$, the sequence $\{d(f^n x, M)\}$ is nonincreasing and that $d(fx, M) < d(x, M)$ if $x \notin M$. We want to show that $d(f^n x, M) \to 0$. Suppose that $d(f^n x, M) \to \varepsilon > 0$. Then for any $\delta > 0$, there exists a number k such that $\varepsilon \leq d(f^k x, M) < \varepsilon + \delta$. By the condition (d), we have $d(f^{k-1} x, M) < \varepsilon$, a contradiction. Consequently, the condition (1) of Theorem B is satisfied.

In order to show the condition (2), choose any positive sequence $\varepsilon_k = b_k \to 0$. Then for each ε_k there exists a $\delta_k > 0$ satisfying the condition (d). Choose a positive sequence $\{a_k\}$ with $b_k < a_k < b_k + \min\{1/k, \delta_k\}$. Then $a_k \to 0$. Suppose $d(x, M) \leq a_k$. Then either (i) $d(x, M) = 0$ or (ii) $0 < d(x, M) < b_k$ or (iii) $b_k \leq d(x, M) \leq a_k$.

Case (i): We have $d(fx, M) = 0$ by the hypothesis.

Case (ii): Since $x \notin M$, we have $d(fx, M) < d(x, M) < b_k$.

Case (iii): Since $\varepsilon_k \leq d(x, M) < \varepsilon_k + \delta_k$, by the condition (d), we have $d(fx, M) < \varepsilon_k = b_k$.

Therefore, in any case, we have $d(fx, M) < b_k$. This shows that (2) holds. This completes our proof.

REMARK. (1) The following example shows that the closedness of G in Theorem 1 is indispensable: $X = \mathbf{R}$, $G = (0, 1]$, $M = \{0\}$, and $f: G \to G$ is given by $fx = x/2$.

(2) The following example shows that the convexity of G in Theorem 1 is indispensable: $X = \mathbf{R}$, $G = [0, 1] \cup [2, 3]$, $M = [1, 2]$, and $f: G \to G$ is given by $fx = 2$ for $x \in [0, 1]$ and $fx = 1$ for $x \in [2, 3]$.

In the following, as in [4], $G \in \mathfrak{F}$ means that G has a locally finite cover by closed convex subsets, and $K \in \mathfrak{F}_0$ means that K is a union of a finite number of closed convex subsets of a Banach space X.

Now we can extend Theorem 1 as follows:

THEOREM 2. *Let X be a Banach space, $G \in \mathfrak{J}$, U an open subset of G, and $f: U \to G$ a continuous map. Assume that :*

(1) *There exists a compact set $M \subset U$ which is an attractor for compact sets under f.*

(2) *There exists a compact set $K \in \mathfrak{J}_0$ such that $M \subset K \subset U$ and such that K is contractible in itself to a point.*

(3) *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$f(N_{\varepsilon+\delta}(M)) \subset N_\varepsilon(M).$$

Then f has a fixed point.

Theorem 2 can be proved by modifying the proof of Nussbaum [6, Theorem 2]. In fact, instead of the condition (3), Nussbaum assumed the following

(3)′ There exists a real-valued function $\rho: [0, a] \to [0, a]$, $(a > 0)$ such that ρ is continuous from the right, $\rho(r) < r$ for $0 < r \leq a$, and $f(N_r(M)) \subset N_{\rho(r)}(M)$ for $0 \leq r \leq a$.

Similarly we can obtain an extended form of Nussbaum [6, Corollary 11].

In order to show our final results, we recall the following generalization of the Frum-Ketkov theorem.

THEOREM C (Buley [2, Corollary 1]) *Let E be a normed linear space and $X \in \mathfrak{J}_0$ be contractible. Suppose $f: X \to E$ is continuous such that $f(\partial X) \subset X$, there is a nonempty compact subset M of E, and*

(a) *there is a constant $c < 1$ such that $d(fx, M) \leq c\, d(x, M)$ for all $x \in X$.*

Then f has a fixed point.

By applying Theorem C, we obtain the following.

THEOREM 3. *Let G be a closed convex subset of a normed linear space E and F be a commutative family of nonexpansive selfmaps of G. Assume there exist an $f \in F$ and a nonempty compact subset M of E such that the condition* (a) *holds. Then there is a point $x \in M$ such that $fx = x$ for all $f \in F$.*

Theorem 3 can be proved by following the proof of Belluce-Kirk [1, Theorem 1] and by applying Theorem C.

Note that if G is a nonempty, bounded, closed, convex subset of a Banach space, then Theorem 3 follows from [1, Theorem 1].

Finally, we obtain

COROLLARY. *Let G be a closed convex subset of a normed linear space E and $f: G \to G$ be a nonexpansive map. Assume there exist a positive integer n and a nonempty compact subset M of E such that f^n satisfies the condition* (a). *Then f has a fixed point.*

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