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A Study on Optimal sampling acceptance plans with respect to a linear loss function and a beta- binomial distribution

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ABSTRACT

We discuss a model for acceptance/rejection decision regarding finite populations. The model is based on a beta-binomial prior distribution and additive costs -- relative sampling costs, relative sorting costs and costs of accepted defectives.

A substantial part of the paper is devoted to constructing a Bayes sequential sampling acceptance plan (BSSAP) for attributes under the model. It is shown that the Bayes fixed size sampling acceptance plans (BFSAP) are better than the Hald's (1960) single sampling acceptance plans based on a uniform prior.

Some tables and examples are provided for comprisons of the minimum Bayes risks of the BSSAP and those of the BFSAP based on a uniform prior and the model.

1. Introduction

In the field of quality control, the sampling inspection plans for attributes play an important role. Dodge and Romig (1959) in their pioneering work introduced the concepts of the acceptable quality level (AQL), the lot tolerance percent defective (LTPD), the average total inspection (ATI), the consumer's risk and the producer's risk. Then they considered a single sampling inspection plan based on the number of defective items in the sample of size n and the acceptance number. In this frame work, the most important and useful concept is the operating characteristic, i.e., the probability of accepting the lot. The basic idea in this frame work is determination of the sample size and the acceptance number to control

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the operating characteristic subject to some additional requirements. Hald (1960), in his fundamental paper, considered Bayesian single sampling inspection plans. In fact, he pointed out, in a logical way, that all sampling inspection plans implicitly assume the existence of a prior distribution and certain costs. In particular, he showed that the AQL concept, a leading concept in some sampling inspection plans, implies both the idea of a prior distribution and some cost considerations. The following remark by him is noteworthy:

It does not pay to inspect if the lots submitted for inspection are produced from a process which is in statistical control with a process p less than or equal to the AQL, i.e., a binomial prior distribution.

Later, Thyregod (1974) considered a generalization of the model by Hald (1960), and derived Bayesian single sampling acceptance plans for the problems admitting a sufficient statistic with a monotone likelihood ratio. Zacks (1970) applied the Bayes sequential methods to the estimation of the number of defectives, and made comparison with the 1-and-2 stage stratified sampling schemes by Grosh (1969).

Section 2 introduces the formulation of the problem along with notations to be used. In Section 3, derivation of the Bayes sequential sampling acceptance plan is discussed, and Section 4 consists of comparisons of minimum Bayes risks of BSSAP's and BFSAP's wrt the uniform prior and a linear cost function.

2. A model of sampling acceptance plans

Let $\{X_1, \ldots, X_N\}$ denote a sequence of dependent Bernoulli trials which arise from sampling a lot of N items without replacement, θ of which are defective and N-θ non-defective. Thus, for any fixed n, $1 \leqslant n \leqslant N$, the probability function of $(X_1, \ldots, X_N)$ is given by

$$(2.1)\quad f(x_1, \cdots, x_n \mid \theta)$$

$$= \binom{\theta}{s_n}\binom{N-\theta}{n-s_n} \Big/ \binom{N}{n}\binom{n}{s_n}$$

where $s_n = x_1 + \cdots + x_n$ denotes the number of defective items in the sample.

Let $k_1 \geq 0$ denote the relative sampling cost per unit item and $k_2 \geq 0$ the relative sorting cost per unit item. Then, the costs can be composed of two parts:

(a) the costs for the accepted lot,
$nk_1 + (\theta - s_n)$, and

(b) the costs for the rejected lot,
$nk_1 + (N-n)k_2$.

This cost structure introduced by Hald (1960) can be written as follows:

(2.2)
$$L(\theta, (n, \delta_n(x_1, \cdots, x_n)))$$
$$= k_1 n + (\theta - s_n)\delta_n(x_1, \cdots, x_n)$$
$$+ k_2(N-n)(1-\delta_n(x_1, \cdots, x_n))$$

where $\delta_n(x_1, \ldots, x_n)$ denotes the probability of accepting the lot after observing
$X_1 = x_1, \ldots, X_n = x_n$.

Now we try to find a rule which minimizes the average loss given by (2.2) wrt the beta-binomial prior $BB(\alpha, \beta, N)$ whose probability function is given by

$$(2.3)\quad g(\theta) = \binom{N}{\theta}\Gamma(\alpha+\beta)\Gamma(\theta+\alpha)\Gamma(N+\beta-\theta)/(\Gamma(\alpha)\Gamma(\beta)\Gamma(N+\alpha+\beta)),$$
$$\theta = 0, 1, \cdots, N.$$

Note that the beta-binomial prior with $\alpha=\beta=1$ is the vague prior, i.e., uniform distribution over the integers 0,1,...,N. also, note that the beta-binomial prior is a conjugate prior so that the posterior distribution of θ, given $X_1 = x_1, \cdots, X_n = x_n$, is given by $BB(\alpha + s_n, \beta + n - s_n, N-n)$, *i.e.*,

$$(2.4)\quad g(\theta \mid X_1 = x_1, \cdots, X_n = x_n)$$
$$= \binom{N-n}{\theta - s_n}\Gamma(\alpha+\beta+n)\Gamma(\theta+\alpha)\Gamma(N+\beta-\theta)/(\Gamma(\alpha+s_n)\Gamma(n+\beta-s_n)\Gamma($$

$N+\alpha+\beta)),$

$\theta-s_n=0,1,\cdots,N.$

3. Bayes sequential sampling acceptance plans

This section describes how the sequential sampling acceptance plan can be determined following the backward induction given in Section 7.2 of Ferguson (1967).

To derive the terminal decision rule, we need the following simple fact.

Lemma 1. Suppose Z has the beta-binomial distribution $BB(\alpha,\beta,N)$.

Then $E((Z+\alpha)(Z+\alpha+1)\cdots(Z+\alpha+k-1))$
$=\alpha(\alpha+1)\cdots(\alpha+k-1)(\alpha+\beta+N)(\alpha+\beta+N+1)\cdots(\alpha+\beta+N+k-1)/((\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1)).$

In parcicular, $E(Z)=N\alpha/(\alpha+\beta)$
$Var(Z)=N\alpha\beta(\alpha+\beta+N)(\alpha+\beta)^{-2}(\alpha+\beta+1)^{-1}.$

It follows from (2.4) and Lemma 1 that the posterior expected loss is given by

$$(3.1)\begin{cases}(N-n)(s_n+\alpha)/(n+\alpha+\beta) & \text{if we accept the lot,}\\ (N-n)k_2 & \text{if we reject the lot,}\end{cases}$$

after observing $X_1=x_1,\cdots,X_n=x_n$.

Next, to determine the stopping rule, we only need to consider the backward induction since the problem is truncated at N. The minimum Bayes conditional expected loss, after observing $X_1=x_1,\cdots,X_n=x_n$, is given by

$$(3.2)\quad U_n(s_n)=nk_1+(N-n)\cdot\min\{k_2,(s_n+\alpha)/(n+\alpha+\beta)\},\quad n=0,1,\cdots,N.$$

Following the notations in Ferguson (1967), let $V_n^{(N)}(n=N,N-1,\ldots,0)$ denote the minimum conditional Bayes risk after observing $X_1=x_1,\cdots,X_n=x_n$, and then $V_0^{(N)}$ is the Bayes risk of the sequential decision problem. Since the conditional distribution of X_{n+1} given $X_1=x_1,\cdots,X_n=x_n$, is given by

$$(3.3)\quad \begin{aligned}P(X_{n+1}=x_{n+1}/X_1=x_1,\cdots,X_n=x_n)&=(\alpha+s_n)/(\alpha+\beta+n)\ \ if\ x_{n+1}=1,\\&=(\beta+n-s_n)/(\alpha+\beta+n)\ \ if\ x_{n+1}=0,\end{aligned}$$

we can express $V_n^{(N)}$ as follows:

$$(3.4)\quad\begin{cases}V_N^{(N)}(s)=U_N(s)=NK_1,\quad s=0,1,\cdots,N\\ V_n^{(N)}(s)=\min\{U_n(s),V_{n+1}^{(N)}(s+1)\cdot((\alpha+s)/(\alpha+\beta+n))+V_{n+1}^{(N)}(s)\cdot((\beta+n-s)/(\alpha+\beta+n))\},\quad s=0,1,\cdots,n,\ n=N-1,\cdots,1,0.\end{cases}$$

Then it follows from Theorem 2 in Section 7.2 of Ferguson (1967) that the Bayes stopping rule is given as follows:

stop sampling after observing $X_1=x_1,\cdots,X_n=x_n$, if and only if

$$U_n(s_n)\le V_{n+1}^{(N)}(s_n+1)\cdot((\alpha+s_n)/(\alpha+\beta+n))+V_{n+1}^{(N)}(s_n)\cdot((\beta+n-s_n)/(\alpha+\beta+n)).$$

Now, to reduce the computational time, we consider an upper bound for truncation. In this regard, we have the next result.

Lemma 2. Suppose that $0<k_2\leqslant k_1<1$. Let J be the smallest integer greater than or equal to $(N+\alpha+\beta)(k_2-k_2^2)/(k_1-k_2^2)-(\alpha+\beta+1)$. Then for all $n\ge J$, it follows that

$$(3.5)\quad U_n(s_n)-E(U_{n+1}(S_{n+1})\ X_1=x_1,\cdots,X_n=x_n)\le 0\quad\text{for all } x_1,\cdots,x_n.$$

It follows from Theorem 5 in Section 7.2 of Ferguson (1967) that the Bayes sequential decision rule truncated at J is the BSSAP for the general problem.

Now, we summarize this section in the following theorem.

Theorem 1. The BSSAP wrt $BB(\alpha,\beta,N)$ and the linear loss function (2.2) is given as follows:

Stop sampling after observing $X_1=x_1,\cdots,X_n=x_n(0\le n\le J)$, if and only if

$$U_n(s_n)\le V_{n+1}^{(J)}(s_n+1)\cdot((\alpha+s_n)/(\alpha+\beta+n))+V_{n+1}^{(J)}(s_n)\cdot((\beta+n-s_n)/(\alpha+\beta+n)),$$

then accept the lot, if and only if

$s_n + \alpha \leq k_2 (n + \alpha + \beta)$.

4. Comparison of minimum Bayes risks of BSSAP's and BFSAP's wrt the uniform prior

In this section Bayes single (fixed size) sampling acceptance plans (BFSAP) will be proposed and compared with the BSSAP's of Section 3 in regard of Bayes risks.

Let $W_n, n=0,1,...,N$, be the Bayes risk of single sampling decision problems based on the sample of size n wrt the uniform prior and (2.2). Then, from (2.3) and (3.2), W_n is expressed by

$$(4.1)\quad W_n = E(U_n(S_n)) = nk_1 + (N-n)(([k_2(2+n)]/(1+n))(([k_2(2+n)]+1)/(2n+4) - k_2) + k_2) \quad \text{if } n > (2k_2-1)/(1-k_2),$$
$$= nk_1 + (N-n)/2 \quad \text{if } n \leq (2k_2-1)/(1-k_2),$$

where [a] is the greatest integer not greater than a.

Table 1 shows the W_n^* s corresponding to $(N, k_1, k_2) = (30, 0.25, 0.25)$. The minimum value of the n-th row of Table 1 of Appendix is equal to W_n for $2 \leq n \leq 30$, but $W_1 < K(1, c)$, where c is the acceptance number. In Table 1 of Appendix if (n, s_n) is on the left hand side of the step-shaped line, we accept the lot from the Bayesian view point (see (3.1)). In fact, $W_n \leq K(n, c)$, for $1 \leq n \leq N$. Therefore, sampling plans using W_n are better than Hald's (1960) optimum sampling plans in aspect of cost saving.

Table 1.

N =30, $k_1 = k_2 = 0.25$

Sample size (n)	Wn
0	7.5
1	7.5
2	7.5
3	7.163
4	7.067
5	7.054
6	7.071
7	7.021
8	7.011
9	7.023
10	7.045
11	7.043
12	7.055
13	7.075
14	7.1
15	7.114
16	7.134
17	7.158
18	7.184
19	7.205
20	7.229
21	7.255
22	7.283
23	7.308
24	7.334
25	7.361
26	7.389
27	7.416
28	7.444
29	7.472
30	7.5

Before turning to comparisons of the two Bayesian sampling plans we need the following lemma.;

Lemma 3. Let J be defined as in Lemma 2. Then

$$\min_{0 \leq n \leq J} \{W_n\} = \min_{0 \leq n \leq N} \{W_n\}$$

Proof. For $J \leq n < N$, it follows that

$$W_{n+1} = E(U_{n+1}(S_{n+1})) = E(E(U_{n+1}(S_{n+1}) | X_1 = x_1, \cdots, X_n = x_n)) \geq E(U_n(S_n)) \quad \text{by Lemma 2,} \quad = W_n.$$

If we denote J_1 by

$$J_1 = \min\{j; j \geq (N+2)(k_2 - k_2^2)/(k_1 - k_2^2) - 3, \ j \text{ is an integer}\},$$

the BSSAP wrt the uniform prior and (2.2) is given, by Theorem 1 of section 3, as follows:

Stop sampling after observing $X_1 = x_1, \cdots, X_n = x_n$ $(0 \leq n \leq J_1)$, if and only if

$$(N-n)\min\{(1+s_n)/(2+n), k_2\} + nk_1 \leq V_{n+1}^{(J_1)}(s_n+1)\cdot(1+s_n)/(2+n) + V_{n+1}^{(J_1)}(s_n)(1+n-s_n)/(2+n),$$

then accept the lot, if and only if

$$1 + s_n \leq k_2(2+n).$$

As an example, for $(N, k_1, k_2) = (30, 0.25, 0.25)$, the continuation sets of the BSSAP are tabulated in Table 2.

$V_0^{(N)}$ is so complicated to find in general form that we will compare it with $\min_{0 \leq n \leq N} \{W_n\}$ for some (N, k_1, k_2)'s, after considering the following trivial cases. First, for $0 < k_2(2+n) < 1$ with $0 < k_2 \leq k_1 < 1$ it follows from (4.1) that

$$W_n = nk_1 + (N-n)k_2 = U_n(s_n).$$

Thus, if J_1 is not greater than $1/k_2 - 2$, we have the following result, by Theorem 1 and Lemma 3:

$$(4.2)\quad \min_{0\leq n\leq J}\{W_n\}=W_0=Nk_2=V_0^{(N)},$$

Next, for $1+n\leq k_2(2+n)$ with $0<k_2\leq k_1<1$, using (3.3), we obtain the inequality (3.5). Thus, if J_1 is not greater than $(2k_2-1)/(1-k_2)$, (4.2) also holds by Theorem 1 and Lemma 3. Combining the above two cases, we have the following result.

Table 2

Continuation sets with $(N, k_1, k_2) = (30, 0.25, 0.25)$ with respect to the Uniform prior and (2.2).

Sample size (n)	0	1	2	3	4	5	6	7	8
number of defective items in the sample of size n	0	0 1	0 1 2	0 1 2 3	0 1 2 3 4	0 1 2 3 4 5	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6

9	10	11	12	13	14	15	16	17	18	19	20
1 2 3 4 5 6	2 3 4 5 6	2 3 4 5 6	2 3 4 5 6	2 3 4 5 6	3 4 5 6	3 4 5 6	3 4 5 6	4 5 6	4 5 6	4 5 6	4 5 6

21	22	23	24	25	26
5 6	5 6	5 6	6	6	6

Lemma 4. If J_1 is not greater than

$$\max\{1/k_2-2,\ (2k_2-1)/(1-k_2)\}$$

with $0<k_2\leq k_1<1$, then (4.2) holds.

Let the absolute saving by the BSSAP vs. the corresponding BFSAP for given (N, k_1, k_2) wrt the uniform prior and (2.2) be denoted by abs. saving (N, k_1, k_2), and abs. saving $(N, k_1, k_2)/V_0^{(N)}\times 100$ by rel. saving (N, k_1, k_2).

Table 3(a) and (b) show that abs. saving (N, k_1, k_2) and rel. saving (N, k_1, k_2), N=70, 100, 200, $k_1=k_2$=0.1, 0.2, 0.25, 0.3 (0.1) 0.9, are similar in shape, that both savings are maximized, as far as Table 3 is concerned, when k_2=0.7, and that they approach zero as k_2 gets close to 0 or 1.

Table 4 (a) and (b) tell stories a little different from those of Table 3 that, as far as Table 4 is concerned, abs. saving (N, k_1, k_2) and rel. saving (N, k_1, k_2), N=30, 70, 100, k_1=0.8, k_2=0.25, 0.3 (0.1) 0.8 do not have the same maximum points of k_2 for each N. Fig. 1(a) and (b) are the graphs of Table 4 (a) and (b), respectively. Although the absolute savings in Table 5 (a) increase as N increases, the relative savings do not. The relative savings are, as far as Table 5(b) is concerned, maximized when N=200.

Fig. 2 has the graphs corresponding to the savings of Table 5.

Subroutines based on (3.4) and (4.1) are written in FORTRAN. All computations for the tables in this paper are carried out on IBM 360/370.

Table 3

(a) Abs. saving (N, k_2, k_2) (b) Rel. saving (N, k_2, k_2)

k_2 \ N	(a) 70	(a) 100	(a) 200	k_2 \ N	(b) 70	(b) 100	(b) 200
0.1	0.027	0.046	0.101	0.1	0.394	0.474	0.526
0.2	0.135	0.201	0.379	0.2	1.04	1.086	1.035
0.25	0.213	0.303	0.564	0.25	1.341	1.348	1.265
0.3	0.306	0.423	0.766	0.3	1.655	1.612	1.474
0.4	0.517	0.709	1.126	0.4	2.22	2.148	1.877
0.5	0.73	0.989	1.674	0.5	2.675	2.559	2.188
0.6	0.963	1.261	2.083	0.6	3.16	2.916	2.432
0.7	1.043	1.434	2,392	0.7	3.172	3.069	2.584
0.8	0.53	1.008	2.334	0.8	1.536	2.057	2.399
0.9	0.0	0.0	0.042	0.9	0.0	0.0	0.042

Table 4

(a) Abs. saving (N, 0.8, k_2)

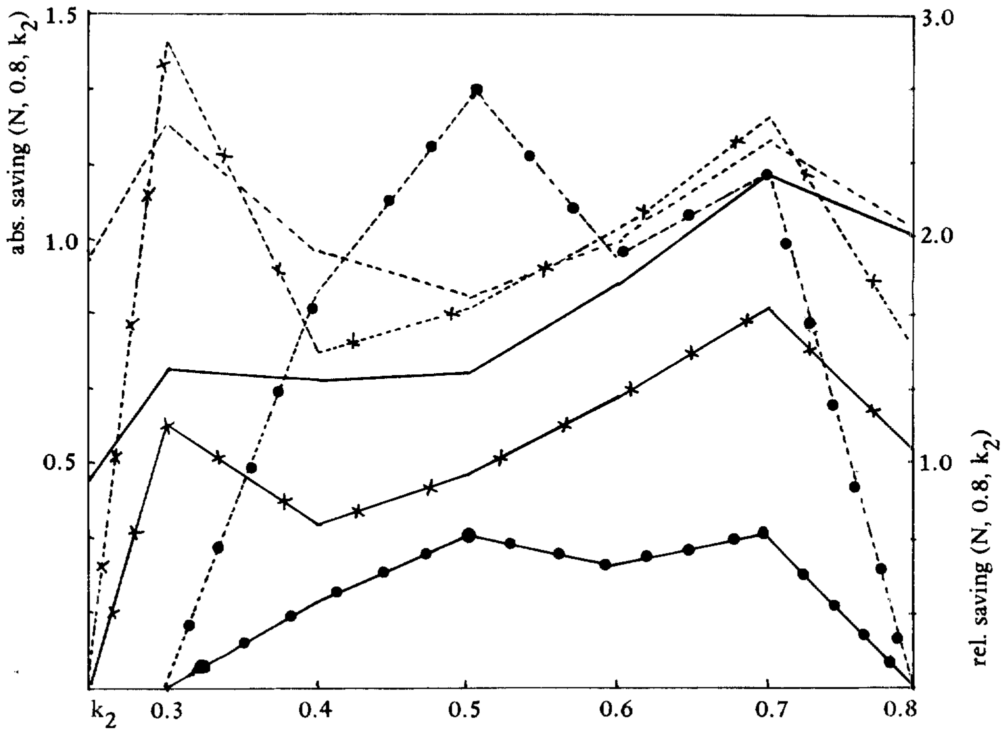
k_2 \ N	30	70	100
0.25	0	0	0.46
0.3	0	0.59	0.71
0.4	0.2	0.37	0.69
0.5	0.342	0.48	0.71
0.6	0.27	0.65	0.9
0.7	0.337	0.85	1.15
0.8	0	0.529	1.01

(b) Rel. saving (N, 0.8, k_2)

k_2 \ N	30	70	100
0.25	0	0	1.89
0.3	0	2.88	2.5
0.4	1.79	1.49	1.95
0.5	2.66	1.69	1.76
0.6	1.93	2.05	2
0.7	2.3	2.55	2.43
0.8	0	1.54	2.06

Fig. 1

(Graph of Table 4)



(a) for abs. saving (30, 0.8, k_2)

for abs. saving (70, 0.8, k_2)

for abs. saving (100, 0.8, k_2)

(b) for rel. saving (30, 0.8, k_2)

for rel. saving (70, 0.8, k_2)

for rel. saving (100, 0.8, k_2)

Table 5.

(a) Abs. saving (N, 0.8, k_2)

k_2 \ N	100	200	300	400	500
0.25	0.46	1.38	1.92	2.62	2.9
0.5	0.71	1.47	2.05	2.59	3.13
0.8	1.01	2.33	3.1	3.82	4.43

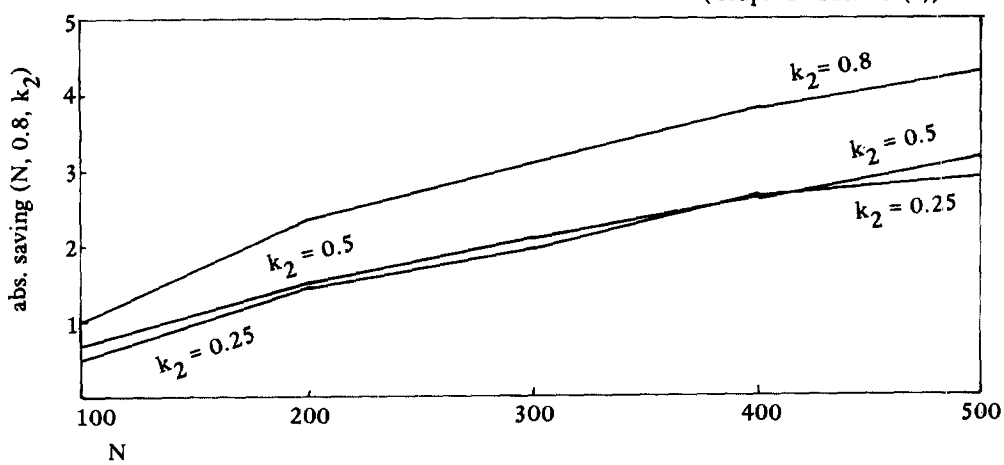
(b) Rel. saving (N, 0.8, k_2)

k_2 \ N	100	200	300	400	500
0.25	1.89	2.9	2.73	2.81	2.5
0.5	1.76	1.86	1.74	1.66	1.61
0.8	2.06	2.4	2.13	1.97	1.83

Fig. 2

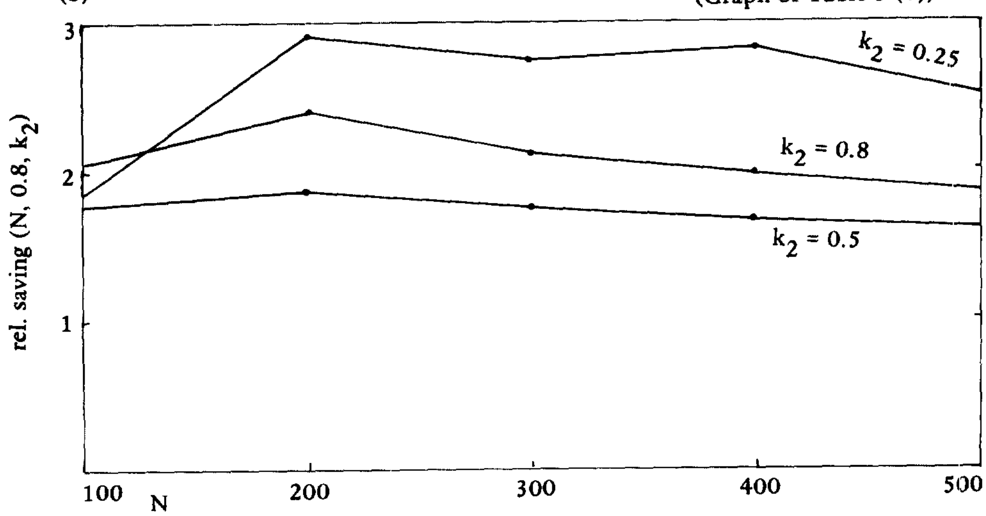
(a)

(Graph of Table 5 (a))



(b)

(Graph of Table 5 (b))



5. Concluding remarks

The approach to the theory of sampling acceptance plans presented here is based directly on the Hypergeometric distribution instead of, as usual, on approximations such as the Binomial or the Poisson distributions.

How much is the BSSAP optimal?
As is illustrated in Section 4, measurement of optimality of the BSSAP vs. the corresponding BFSAP is tried in this paper, and for such cases as $(N, k_1, k_2) = (70, 0.6, 0.6)$, $(70, 0.7, 0.7)$ and $(100, 0.7, 0.7)$ the relative savings are as much as more than 3% (see Table 3(b)). But this paper failed to derive a general method of finding the optimal points (N^*, k_1^*, k_2^*) subject to some conditions on them, and this deserves further studying on to improve our sampling plans in the field of Quality Control.

References

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국 문 요 약

로트중의 결함품목의 수에 대하여 비복원추출법에 의한 표본검사를 실시할 때, Hald(1960)의 손실함수와 beta-binomial 사전분포를 고려한 Bayes 축차표본조사계획을 유도하고, Bayes 축차표본조사계획과 Hald (1960)의 단순표본조사계획을 비교 연구하였다.

APPENDIX

(extracted from p.316 of Ref. (4))

Table 1

Values of K(n,c) for N = 30 and ks = kr = 1/4

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	8.71	14.75																													
2	7.50	9.83	14.50																												
3	7.16	8.18	10.54	14.25																											
4	7.07	7.50	8.80	10.97	14.00																										
5	7.05	7.20	7.95	9.29	11.22	13.75																									
6	7.07	7.07	7.50	8.36	9.64	11.36	13.50																								
7	7.10	7.02	7.26	7.82	8.70	9.90	11.41	13.25																							
8	7.13	7.01	7.13	7.50	8.11	8.97	10.07	11.41	13.00																						
9	7.17	7.02	7.07	7.31	7.74	8.36	9.17	10.17	11.37	12.75																					
10	7.20	7.05	7.05	7.20	7.50	7.95	8.56	9.32	10.23	11.29	12.50																				
11	7.23	7.07	7.04	7.13	7.35	7.68	8.14	8.72	9.42	10.24	11.18	12.25																			
12	7.25	7.11	7.06	7.11	7.26	7.51	7.85	8.30	8.84	9.48	10.23	11.07	12.00																		
13	7.28	7.14	7.07	7.10	7.20	7.38	7.64	7.99	8.41	8.92	9.50	10.17	10.92	11.75																	
14	7.30	7.17	7.10	7.10	7.17	7.30	7.50	7.77	8.10	8.50	8.97	9.50	10.10	10.77	11.50																
15	7.32	7.20	7.13	7.12	7.16	7.25	7.40	7.61	7.87	8.19	8.56	8.99	9.47	10.01	10.80	11.25															
16	7.3	7.23	7.16	7.13	7.16	7.23	7.34	7.50	7.71	7.96	8.25	8.60	8.99	9.42	9.90	10.43	11.00														
17	7.30	7.25	7.19	7.16	7.17	7.21	7.30	7.42	7.58	7.78	8.02	8.30	8.61	8.96	9.35	9.78	10.25	10.75													
18	7.37	7.28	7.22	7.18	7.18	7.22	7.28	7.37	7.50	7.66	7.85	8.07	8.32	8.61	8.92	9.27	9.65	10.06	10.50												
19	7.39	7.30	7.24	7.21	7.21	7.23	7.27	7.34	7.44	7.57	7.72	7.89	8.10	8.33	8.58	8.86	9.17	9.50	9.86	10.25											
20	7.40	7.33	7.27	7.24	7.23	7.24	7.27	7.33	7.40	7.50	7.62	7.76	7.92	8.11	8.31	8.54	8.79	9.06	9.35	9.67	10.00										
21	7.42	7.35	7.30	7.27	7.26	7.26	7.28	7.32	7.38	7.46	7.55	7.66	7.79	7.94	8.10	8.28	8.48	8.70	8.94	9.19	9.46	9.75									
22	7.43	7.37	7.33	7.30	7.28	7.28	7.30	7.33	7.37	7.43	7.50	7.59	7.69	7.80	7.94	8.08	8.24	8.41	8.60	8.80	9.02	9.25	9.50								
23	7.44	7.39	7.35	7.33	7.31	7.31	7.32	7.34	7.37	7.41	7.47	7.54	7.61	7.70	7.81	7.92	8.05	8.18	8.33	8.49	8.66	8.85	9.04	9.25							
24	7.45	7.41	7.38	7.35	7.34	7.33	7.34	7.35	7.38	7.41	7.45	7.50	7.56	7.63	7.71	7.80	7.89	8.00	8.11	8.37	8.37	8.52	8.67	8.83	9.00						
25	7.46	7.43	7.40	7.38	7.37	7.36	7.36	7.37	7.41	7.44	7.48	7.48	7.52	7.57	7.63	7.70	7.77	7.85	7.94	8.03	8.14	8.24	8.36	8.61	.86	8.75					
26	7.46	7.44	7.40	7.41	7.39	7.36	7.36	7.37	7.41	7.41	7.44	7.48	7.52	7.57	7.63	7.70	7.77	7.85	7.94	8.	7.95	8.02	8.11	8.20	8.29	8.40	8.50				
27	7.48	7.46	7.44	7.43	7.42	7.42	7.42	7.42	7.43	7.44	7.45	7.47	7.49	7.51	7.54	7.57	7.61	7.65	7.69	7.74	7.79	7.84	7.90	7.96	8.03	8.10	8.17	8.25			
28	7.49	7.47	7.46	7.45	7.45	7.45	7.44	7.45	7.45	7.45	7.46	7.47	7.49	7.50	7.52	7.54	7.56	7.58	7.61	7.64	7.67	7.70	7.74	7.78	7.82	7.86	7.91	7.95	8.00		
29	7.49	7.49	7.48	7.48	7.47	7.47	7.47	7.47	7.47	7.48	7.48	7.49	7.49	7.50	7.51	7.51	7.52	7.54	7.55	7.56	7.58	7.59	7.61	7.63	7.64	7.66	7.69	7.71	7.73	7.75	
30	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50	7.50