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Confidence Intervals on Variance Components in Two-Way Classification with Interaction Model

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ABSTRACT

Arvesen (1969) has shown a procedure which produces an approximate confidence interval for a variance component in unbalanced one-way classification model. In this paper, his work is extended to the case when the model of interest is unbalanced two-way classification. Following the procedure described in this paper, approximate confidence intervals are computed by a Monte Carlo simulation.

1. Introduction

The problem discussed in this paper is that of obtaining confidence intervals for variance components in the unbalanced two-way classification with interaction model which is given by

$$Y_{ijk}=\mu+\alpha_i+\beta_j+\gamma_{ij}+e_{ijk} \quad \cdots\cdots\cdots (1)$$
$$1=1,\cdots l, \quad j=1,\cdots,m, \quad k=1,\cdots, n_{ij}$$

where

μ is an unknown constant, $\{\alpha_i\}$, $\{\beta_j\}$, $\{\gamma_{ij}\}$, and $\{e_{ijk}\}$ are all mutually independent normal random variables with zero means and variances σ^2_α, σ^2_β, σ^2_γ, and σ^2_e, respectiviely. We assume that $n_{ij}\geqq 2$ and without loss of generality $\ell \geqq m$. In Section 2, formulas are given for variance component estimators derived by the similar methods in Arvesen (1969). Section 3 discusses how confidence intervals are obtained for the variance components considered in Section 2. In Section 4, several methods which produce an unbiased estimator for a variance component are introduced and are compared with the estimator based on the procedure illustrated in Section 2 using Monte Carlo simulation.

2. Estimators

We begin by defining a two-sample U-statistic Let $X_1,\cdots,X_\ell$ be independent, identically distributed random variables, let $Y_1,\dots, Y_m$ be m independent, identically distributed random variables, and let Y'_s be independent. A parameter η is said to be estimable of degree (r, s), if r and s are the smallest sample sizes for which there exists an estimator of η that is unbiased. That is, there is a function $h^*(.)$ such that

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$$E[h^*(X_1, \cdots, X_r; \quad Y_1, \cdots Y_s)] = \eta$$

without loss of generality, the above function $h^*(.)$ can be assumed to be symmetric in its X_i components and separately symmetric in its Y_j components. Let $h(.)$ denote such a symmetric function and name it a symmetric kernel. Then we define that a two-sample U-statistic for the estimable parameter η of degree$(r, s)$with the symmetric kernel $h(.)$ has the form

$$U(X_1, \cdots, X_l; \ Y_1, \cdots, Y_m)$$
$$=\binom{\ell}{r}^{-1}\binom{m}{s}^{-1} \sum_{\alpha\in A} \sum_{\beta\in B} h(X_{\alpha 1}, \cdots, X_{\alpha r}; \ Y_{\beta 1}, \cdots, Y_{\beta s})$$

where A (or B)is the collection of all subsets of r (or s) integers chosen without replacement from the integers $\{1, \cdots, \ell\}$
(or$\{1, \cdots, m\}$)(See Randles and Wolfe (1979)).

Arvesen (1969) considered that obtaining robust procedures in the model such that (1) can be done by studying U-statistics, and functions of several U-statistics. Doing this, he modified the model (1) as follows

$$Y_{ijk} = \mu + \alpha(U_i) + \beta(V_j) + \gamma(U_i, V_j) + e(U_i, \ V_j)_k \tag{2}$$

$$1 = 1, \cdots, \ell, \quad j = 1, \quad \cdots, m, \quad k = 1, \cdots, n_{ij}$$

where

the $U_i' s$ $i=1, \ldots, \ell$, are an independent sample from some infinite population,

(b) the V_j' $s, j=1, \ldots,$ m, are an independent sample from some infinite population, and

(c) the two populations are independently sampled(U and V_j are independent for all $(i, \ j)$, and the meanings of α, β, γ, e,and μ remain all the same as in the model (1).

Let $h_1(U_{\alpha_1}, \ U_{\alpha_2}; \ V_{\beta_1}, \ V_{\beta_2})$
$$=[(Y_{\alpha_1\beta_1} - Y_{\alpha_1\beta_2} - Y_{\alpha_2\beta_1} + Y_{\alpha_2\beta_2})^2 - (n^{-1}_{\alpha_1\beta_1} + n^{-1}_{\alpha_1\beta_2} + n^{-1}_{\alpha_2\beta_1} + n^{-1}_{\alpha_2\beta_2})\sigma_e^2] \ / \ 4$$

be a symmetric kernel with U-"statistic"

$$U^{(1)} = \binom{\ell}{2}^{-1}\binom{m}{2}^{-1}\{\sum_\alpha \sum_\beta h_1(U_{\alpha_1}, U_{\alpha_2}; \ V_{\beta_1}, V_{\beta_2})\}$$
$$= (\ell-1)^{-1}(m-1)^{-1}\{\sum_i \sum_j (Y_{ij} - Y_{i\cdot} - Y_{\cdot j} + Y_{\cdot\cdot})^2\} - \ell^{-1} m^{-1} (\sum_i \sum_j n_{ij}^{-1}) \sigma_e^2$$

where $Y_{ij} = n_{ij}^{-1} \sum_k Y_{ijk}$, $Y_{i\cdot} = m^{-1} \sum_j Y_{ij}$,
$Y_{\cdot j} = \ell^{-1} \sum_i Y_{ij}$, and
$Y_{\cdot\cdot} = \ell^{-1} m^{-1} \sum \sum_j Y_{ij}$

Let $h_2(Y_{\alpha_1}; V_{\beta_1})$
$$= [(n_{\alpha_1\beta_1} - 1)^{-1} \sum_k (Y_{\alpha_1\beta_1 k} - Y_{\alpha_1\beta_1})^2 - \sigma_e^2]$$

be a symmetric kernel with U-"statistic"

$$U^{(2)} = \binom{\ell}{1}^{-1}\binom{m}{1}^{-1} \sum_\alpha \sum_\beta h_2(U_{\alpha_1}; \ V_{\beta_1})$$
$$= \ell^{-1} m^{-1} \{\sum \sum (n_{ij} - 1)^{-1} \sum_k (Y_{ijk} - Y_{ij})^2\} - \sigma_e^2.$$

Note that $U^{(1)}$ and $U^{(2)}$ are not statistics since they depend on the unknown σ_e^2. But when we let

$$g(U^{(1)}, \ U^{(2)}) = U^{(1)} - K_{lm} U^{(2)},$$
$$K_{lm} = \ell^{-1} m^{-1} (\sum \sum n_{ij}^{-1}),$$

then $g(U^{(1)}; U^{(2)})$is a statistic obviously.Furthermore,we can show that the expectation of $g(U^{(1)}, \ U^{(2)})$ is equal to σ_r^2.
That is,$g(U^{(1)}, U^{(2)})$ is an unbiased estimator of σ_γ^2.

Let $h_3(U_1, \cdots, U_l; \ V_{\beta_1}, \ V_{\beta_2})$
$$= [(\ell^{-1} \sum_i Y_{i\beta_1} - \ell^{-1} \sum_i Y_{i\beta_2})^2 - \ell^{-1}\sigma_r^2 - \ell^{-2} \sum (n^{-1}_{i\beta_1} + n^{-1}_{i\beta_2}) \sigma_e^2] \ / \ 2$$

be a symmetric kernel with U-"statistic"

$$U^{(3)} = \binom{\ell}{\ell}^{-1}\binom{m}{2} \sum_\alpha \sum_\beta h_3(U_1, \cdots, U_l; \ V_{\beta_1}, \ V_{\beta_2})$$
$$= (m-1)^{-1} \{\sum (Y_{\cdot j} - Y_{\cdot\cdot})^2\} - \ell^{-1}\sigma_r^2 - \ell^{-1} K_{lm} \sigma_e^2$$

Also note that $U^{(3)}$ is not a statistic, but when we define

$$g(U^{(1)},\ U^{(2)},\ U^{(3)}) = (m-1)^{-1}\{\sum_j (Y_{\cdot j}-Y_{\cdot\cdot})^2\} - \ell^{-1}(\ell-1)^{-1}(m-1)^{-1}\{\sum_i\sum_j (Y_{ij}-Y_{i\cdot}-Y_{\cdot j}+Y_{\cdot\cdot})^2\}$$

then $g(U^{(1)}, U^{(2)}, U^{(3)})$ is a statistic whose expectation is equal to σ_β^2. That is $g(U^{(1)}, U^{(2)}, U^{(3)})$ is an unbiased estimator of σ_β^2.

Let
$$h_4(U_{\alpha_1},\ U_{\alpha_2};\ V_1,\ \cdots,\ V_m) = [(m^{-1}\sum_j Y_{\alpha_1 j} - m^{-1}\sum_j Y_{\alpha_2 j})^2 - m^{-1}\sigma_\gamma^2 - m^{-2}\sum_j (n_{\alpha_1 j}^{-1}+n_{\alpha_2 j}^{-1})\sigma_e^2]\ /\ 2$$

be a symmetric kernel with U-"statistic"

$$U^{(4)} = \binom{\ell}{2}^{-1}\sum_\alpha\sum_\beta h_4(U_{\alpha_1}, U_{\alpha_2};\ V_1,\ \cdots,\ V_m) = (\ell-1)^{-1}\{\sum (Y_{i\cdot}-Y_{\cdot\cdot})^2\} - m^{-1}\sigma_\gamma^2 - m^{-1}K_{\ell m}\sigma_e^2$$

Note also that $U^{(4)}$ is not a statistic, but when we define

$$g(U^{(1)},\ U^{(2)},\ U^{(4)}) = (\ell-1)^{-1}\{\sum_i (Y_{i\cdot}-Y_{\cdot\cdot})^2\} - m^{-1}(m-1)^{-1}(\ell-1)^{-1}\{\sum_i\sum_j (Y_{ij}-Y_{i\cdot}-Y_{\cdot j}+Y_{\cdot\cdot})^2\}$$

then $g(U^{(1)}, U^{(2)}, U^{(4)})$ is a statistic and can be an unbiased estimator of σ_α^2 since its expectation is equal to σ_α^2.

3. Jackknife Estimate

In this section, we describe the jackknife procedure for two-sample problems introduced in Arvesen (1969).

Let $X_1, \ldots, X_L$ be a random sample from the first population and $Y_1, \ldots, Y_M$ from the second population. Let $L=\ell k$, $M=mh$ (all integers) and split the X's into ℓ groups of k observations each, and the Y's into m groups of h observations each. Let $\hat\theta_0$ be the estimate of θ based on all the observations, and let $\hat\theta_{i\cdot}$ denote the estimate obtained after deletion of the ith group of X's, $i=1,\ldots,\ell$, and let $\hat\theta_{\cdot j}$ denote the estimate obtained after deletion of the jth group of Y's, $j=1,\ldots, m$. Next,

Let
$$\hat\theta_{i\cdot}^p = \ell\hat\theta_0 - (\ell-1)\hat\theta_{i\cdot}^p,\quad i=1,\ \cdots,\ \ell,$$
$$\hat\theta_{\cdot j}^p = m\hat\theta_0 - (m-1)\hat\theta_{\cdot j},\quad j=1,\cdots,m,$$

and define the jackknife estimate of θ to be

$$\hat\theta = (\ell+m)^{-1}(\sum_i \hat\theta_{i\cdot}^p + \sum_j \hat\theta_{\cdot j}^p)$$

Let
$$\hat\theta_\ell = \ell^{-1}\sum_i \hat\theta_{i\cdot}^p,\quad \hat\theta_m = m^{-1}\sum_j \hat\theta_{\cdot j}^p,\quad \text{and}$$
$$S_g^2 = m^{-1}\{\ell^{-1}(\ell-1)^{-1}\sum_i (\hat\theta_{i\cdot}^p - \hat\theta_\ell)^2 + m^{-1}(m-1)^{-1}\sum_j (\hat\theta_{\cdot j}^p - \hat\theta_m)^2\}.$$

Then, if m remains finite as $\sum_i\sum_j n_{ij}\to\infty$, Theorem 1 and 2 of Arvesen and Layard (1975) can be applied to the estimators derived in Section 2. That is, the conjecture

$$m^{1/2}(\hat\theta-\theta)\ /\ S_g \to \pounds\, t_{m-1} \quad \text{as} \quad \sum_i\sum_j n_{ij}\to\infty$$

is valid, for example, with

$$\hat\theta_0 = g(U^{(1)},\ U^{(2)}),\quad \theta = \sigma_\gamma^2.$$

4. Numerical Example

To examine the validity of the procedure described above, confidence intervals are calculated. The observations are generated by GAUSS (one of the subroutines in SSP) and DOUEXP. The model considered is :

$$Y_{ijk} = \alpha_i + \beta_j + \gamma_{ij} + e_{ijk},\quad i=1,\ \cdots,\ 10,$$
$$j=1,\cdots,5,\quad k=1,\cdots,n_{ij},\quad n_{i1}=2,$$
$$n_{i2}=n_{i3}=n_{i4}=3,\quad n_{i5}=4.$$

The $\{\alpha_i\}$, $\{\beta_j\}$, $\{\gamma_{ij}\}$ and $\{e_{ijk}\}$ are mutually independent random variables with mean zero, and variances σ_α^2, σ_β^2, σ_γ^2, σ_e^2, respectively. Two kinds od distributions, normal and double exponential, are considered for the $\{\alpha_i\}$, $\{\beta_j\}$, $\{\gamma_{ij}\}$ and $\{e_{ijk}\}$ with $\sigma_\alpha^2=4$, $\sigma_\beta^2=9$, $\sigma_\gamma^2=81$ and $\sigma_e^2=25$

Note that the estimators obtained in Section 2 have forms similar to those of unweighted means analysis method, but computation of them are rather simple. So we apply the jackknife technique to the estimators considered in Section 2. For the purpose of comparison let us introduce some other forms of estimators drawn from the methods such as Henderson's methods I and III, weighted means analysis method. For a detailed, illustration the reader is referred to Harville (1967), and Searle (1971). Now let

$$N_{i.}=\sum_j n_{ij},\quad N_{.j}=\sum_i n_{ij},\quad N=\sum_i\sum_j n_{ij},\qquad RT=\sum_i\sum_j\sum_k \bar{Y}^2_{ijk},$$
$$R(\mu)=N\bar{Y}^{-2}\cdots,\qquad R(\mu,\alpha)=\sum N_{i.}\bar{Y}^{-2}_{i..},$$

$$R(\mu,\beta)=\sum N_{.j}\bar{Y}^{-2}_{.j.},\quad \text{and}\quad R(\mu,\alpha,\beta,\gamma)=\sum\sum n_{ij}\bar{Y}^2_{ij..}$$

Let the $m\times 1$ vector $\hat{\beta}$ to be any solution to $C_\beta=q$, where C is a $m\times m$ matrix with rank r and its elements are

$$C_{ij}=N_{.j}-\sum_i\left(\frac{n^2_{ij}}{N_{i.}}\right),\qquad j=1,\cdots,m,$$

$$C_{jh}=-\sum_i\left(\frac{n_{ij}\,n_{ih}}{N_{i.}}\right),\quad j\neq h=1,\quad\cdots,m,$$

and the elements of the $m\times 1$ vector q are

$$q_j=Y_{.j.}-\sum_i n_{ij}\bar{Y}_{i..},\qquad \text{and}\qquad R(\mu,\alpha,\beta)$$
$$=R(\mu,\alpha)+\hat{\beta}_q.$$

Then

(a) the Henderson's method 1 estimators have the form

$$\begin{pmatrix}\hat{\sigma}^2_\alpha\\ \hat{\sigma}^2_\beta\\ \hat{\sigma}^2_\gamma\end{pmatrix}=\begin{bmatrix}N-k_3, & k_1-k_4, & k_1-k_5\\ k_2-k_3, & N-k_4, & k_2-k_5\\ k_3-k_2, & k_4-k_1, & N-k_1-k_2+k_5\end{bmatrix}^{-1}$$

$$\begin{bmatrix}SSA-(\ell-1)\,\hat{\sigma}^2_e\\ SSB-(m-1)\,\hat{\sigma}^2_e\\ SSAB-(\ell-1)\,(m-1)\,\hat{\sigma}^2_e\end{bmatrix}$$

where

$$\hat{\sigma}^2_e=\frac{SSE}{(N-m)},\quad k_1=\sum_i\sum_j\left(\frac{n^2_{ij}}{N_{i.}}\right),\quad k_2=\sum_i\sum_j\left(\frac{n^2_{ij}}{N_{.j}}\right),$$

$$k_3=\sum_i\left(\frac{N^2_{i.}}{N}\right),\quad k_4=\sum_j\left(\frac{N^2_{.j}}{N}\right),\quad k_5=\sum_i\sum_j\left(\frac{n^2_{ij}}{N}\right),$$

$$SSA=R(\mu,\alpha)-R(\mu),\quad SSB=R(\mu,\beta)-R(\mu),$$
$$SSAB=R(\mu,\alpha,\beta,\gamma)-R(\mu,\alpha)-R(\mu,\beta)+R(\mu),$$
$$SSE=RT-R(\mu,\alpha,\beta,\gamma)$$

(b) the Henderson's method 3 estimators have the form

$$\hat{\hat{\sigma}}^2_\gamma=\frac{\{SSAB-(\ell-1)\ (m-1)\ \hat{\sigma}^2_e\}}{(N-k_6-k_1)}$$
$$\hat{\hat{\sigma}}^2_\alpha=\frac{\{SSA-(k_6+k_1-k_2)\ \hat{\hat{\sigma}}^2_\gamma-(\ell-1)\ \hat{\sigma}^2_e\}}{(N-k_2)}$$
$$\hat{\hat{\sigma}}^2_\beta=\frac{\{SSB-k_6\ \hat{\hat{\sigma}}^2_\gamma-(m-1)\ \hat{\sigma}^2_e\}}{(N-k_1)}$$

where

$$k_6=k^*-k_1,\quad k^*=\sum_i\sum_j\left(\frac{n^2_{ij}}{N_{i.}}\right)+t_r\,(C_i^{-1}F_i),\ \text{where}$$

$$F_i=\{f_i,\ j\,k\},\quad f_{i,jj}=\left(\frac{n^2_{ij}}{N_{i.}}\right)\left(\frac{\sum_j n^2_{ij}}{N_{i.}+N_{i.}-2n_{ij}}\right)$$

for $\quad j=1,\cdots,m,\ f_{i,jh}$

$$=\left(\frac{n_{ij}n_{ih}}{N_{i.}}\right)\left(\frac{\sum_j n^2_{ij}}{N_{i.}-n_{ij}-n_{ij}}\right)$$

for $\qquad j\neq h=1,\ \cdots,m,$

$$SSA=R(\mu,\alpha,\beta)-R(\mu,\beta),$$
$$SSB=R(\mu,\alpha,\beta)-R(\mu,\alpha),$$
$$SSAB=R(\mu,\alpha,\beta,\gamma)-R(\mu,\alpha,\beta).$$

(c) the weighted means analysis estimators have the form

$$\tilde{\sigma}^2_\gamma=\frac{\{SSAB-(\ell-1)\ (m-1)\ \hat{\sigma}^2_e\}}{(N-k_6-k_1)}=\hat{\hat{\sigma}}^2_\gamma$$
$$\tilde{\sigma}^2_\alpha=\left\{\frac{SSA-k_7\ \tilde{\sigma}^2_\gamma}{m-(\ell-1)\quad\hat{\sigma}^2_e}\right\}/\ k_n$$
$$\tilde{\sigma}^2_\beta=\left\{\frac{SSB-k_8\ \tilde{\sigma}^2_\gamma}{\ell-(m-1)\quad\hat{\sigma}^2_e}\right\}/\ k_8$$

where

$$k_7=\frac{\sum_i W_i-\sum_i W_i^2}{\sum_i W_i},\qquad W_i=\frac{m^2}{(\sum_j n_{ij}^{-1})},$$

$$k_8=\frac{\sum_j V_j-\sum_j V_j^2}{\sum_j V_j},\qquad V_j=\frac{\ell^2}{(\sum_i n_{ij}^{-1})},$$

$$SSA=\sum_i W_i\,(Y_{i..}-Y_w)^2,\ Y_w=\frac{\sum_i W_i\,Y_{i..}}{\sum_i W_i},$$

$$SSB=\sum_j V_j\,(Y_{.j}-Y_v)^2,\qquad Y_v=\frac{\sum_j V_j\,Y_{.j}}{\sum_j V_j},$$

$$SSAB=R(\mu,\alpha,\beta,\gamma)-R(\mu,\alpha,\beta).$$

(d) the Jackkfite estimators

When we denote $\tilde{\tilde{\sigma}}_\alpha^2=g(U^{(1)},U^{(2)},U^{(4)})$, $\tilde{\tilde{\sigma}}_\beta^2=g(U^{(1)},U^{(2)},U^{(3)})$, $\tilde{\tilde{\sigma}}_\gamma^2=g(U^{(1)},U^{(2)})$ as in Section 2, the results of Monte Carlo simulation are shown in Tables 1 and 2. We apply the jackknife technique only to the variance component σ_γ^2 for computational convenience.

5. Concluding Remarks

For the two-way unbalanced classification model we obtained estimators of variance components which are unbiased by introducing U-statistic. We applied the jackknife technique to them in order to produce approximate confidence intervals for the variance components of interest. Table-1 shows that

1. the sample mean of $\tilde{\tilde{\sigma}}_\gamma^2$, the jackknife estimates, is the closest value to the true value $\sigma_\gamma^2=81$.
2. the estimate $\tilde{\tilde{\sigma}}_\gamma^2$ has relatively a small sample variance.
3. the jackknife method gives reasonable confidence intervals.

However, for the non-normal case as shown in Table 2, the jackknife method is not very good in the sense of sample mean. But the sample variance is relatively small compared with the other methods. Table 2 shows that there remain some problems to be investigated for robustness.

Table 1. Monte Carlo simulation for normal distribution

$r_{ij} \sim N(0,\ 81)$: $\sigma_\gamma^2=81$				
a	b and c	d	approximate 95% confidence interval from jackknifing method	
$\hat{\sigma}_\gamma^2$	$\hat{\hat{\sigma}}_\gamma^2 = \tilde{\sigma}_\gamma^2$	$\tilde{\tilde{\sigma}}_\gamma^2$		
85.85	95.44	75.93	(66.18, 85.70)	
69.83	48.45	65.79	(58.07, 73.46)	
72.18	66.89	57.43	(38.16, 76.70)	
107.16	70.38	106.35	(83.88, 128.79)	
87.90	47.59	73.18	(61.27, 85.10)	
104.76	57.60	97.09	(79.23, 114.96)	
49.06	11.94	44.48	(34.12, 54.98)	
110.32	−25.75	77.45	(66.55, 88.36)	
118.27	81.87	119.78	(95.40, 144.12)	
74.67	38.48	69.54	(55.23, 83.77)	
131.45	32.22	76.64	(53.68, 99.59)	
67.47	12.50	57.01	(35.16, 78.78)	
110.62	87.76	80.30	(70.48, 108.18)	
116.59	114.67	93.30	(75.62, 110.96)	
84.14	51.27	83.18	(62.62, 103.73)	
92.68	52.69	79.10	:	sample mean
547.55	1307.16	394.37	:	sample variance

Table 2. Monte Carlo simulation for double exponential distribution.

$r_{ij} \sim$ DOUEXP(0, 81) : $\sigma_Y^2=81$			
a $\dot{\sigma}_Y^2$	b and c $\hat{\hat{\sigma}}_Y^2 = \tilde{\sigma}_Y^2$	d $\tilde{\tilde{\sigma}}_Y^2$	approximate 95% confidence intervals from jackknifing method
254.57	98.80	187.18	(130.75, 247.71)
168.56	113.74	154.98	(119.52, 190.41)
155.40	189.02	94.87	(66.49, 123.30)
228.17	70.50	226.88	(146.17, 308.00)
150.57	6.65	122.50	(109.22, 135.68)
238.26	72.92	198.93	(155.40, 242.49)
383.03	288.68	329.84	(245.43, 414.19)
209.02	162.84	193.42	(132.30, 254.41)
245.34	31.61	126.21	(89.21, 163.56)
117.26	34.55	107.68	(88.50, 126.63)
160.00	124.02	136.60	(111.20, 162.04)
261.42	88.30	236.17	(180.21, 292.01)
186.11	127.71	135.03	(99.30, 170.63)
182.88	118.32	183.54	(133.23, 233.60)
112.63	69.20	83.03	(59.59, 106.35)
203.55	106.46	167.79	: sample mean
4745.88	4934.50	4198.81	: sample variance

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