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DOUBLE B-CENTRALIZERS OF PRE-HILBERT B-MODULES

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1. Introduction

W.L. Paschke [1] investigated right modules over a B^*-algebra B which possess a B-valued inner product respecting the module action. We show that pre-Hilbert B-module X and double B-centralizers $M(X, B)$ are isomorphic as pre-Hilbert B-modules. Moreover, it is easy to see that pre-Hilbert B-module X is a self-dual if and only if every map T in X' has a bounded module map T^* such that $(T, T^*)\in M(X, B)$.

2. Results

Throughout this paper, B will be a B^*-algebra with a multiplicative identity e. All algebras have the complex field $\mathbf{C}$. We denote that X' is the set of all bounded module maps from X to B and denote that B' is the set of all bounded module maps of B into X.

DEFINITION 2.1. A *pre-Hilbert B-module* is a right B-module X equipped with a conjugate-bilinear $<,> : X\times X \longrightarrow B$ satisfying;

(i) $<x, x>\geq 0$, $\forall x\in X$

(ii) $<x, x>=0$ only if $x=0$

(iii) $<xb, y>=<x, y>b$, $\forall x, y\in X$, $b\in B$.

The map $<,>$ will be called a *B-valued inner product* on X.

REMARK. (1) It is easy to see that $<x, y>=<y, x>^*$ from (i).

(2) $<x, yb>=b^*<x, y>$ $\forall x, y\in X$, $b\in B$.

For a pre-Hilbert B-module X, define $\| \ \|_X$ on X by

$$\|x\|_X=\|<x, x>\|^{\frac{1}{2}}$$

PROPOSITION 2.2. (1) $\|x\|_X=\sup\{\|<x, y>\| : y\in X, \ \|y\|_X\leq 1\}$

(2) $\|x\|_X=\sup \ \{\|<xb, y>\| : b\in B, \ y\in X, \ \|b\|\leq 1, \ \|y\|_X\leq 1\}$

PROOF. (1) By [1]. Proposition 2. 3, it is clear.

(2) $\|<xb, y>\|\leq\|xb\|_X\|y\|_X\leq\|x\|_X\|b\|\|y\|_X\leq\|x\|_X$. Since $<xb, y>=<x, y>b$,

$$\sup\{\|<xb, y>\| : b\in B,\ y\in X,\ \|b\|\leq 1,\ \|y\|_X\leq 1\}$$
$$\geq \sup\{\|<x, y>e\| : y\in X,\ \|y\|_X\leq 1\}$$
$$=\|x\|_X$$

DEFINITION 2.3. A pre-Hilbert B-module X which is complete with respect to $\| \|_X$ will be called a *Hilbert B-module.*

The following proposition can be proved by the similar method given in [3, Proposition 5].

PROPOSITION 2.4. *Let X and Y be Hilbert B-modules and let $T' : X\longrightarrow Y$ be a function. If there is a function $T'' : Y\longrightarrow X$ such that*

$$<T'x, y>_Y=<x, T''y>_X$$

then T' and T'' are bounded module maps.

DEFINITION 2.5. By a *double B-centralizer*, we mean a pair (T', T'') such that

$$b^*T'x=<x, T''b>_X$$

for $x\in X$, $b\in B$, $T'\in X'$, and $T''\in B'$.

REMARK. In general, B becomes a Hilbert B-module when we define $<, >$ by $<x, y> =y^*x$ for $x, y\in B$. Therefore, in the Definition 2.5., $b^*T'x$ means $<T'x, b>_B$. i.e. $b^*T'x=<x, T''b>_X$ means $<T'x, b>_B=<x, T''b>_X$.

NOTATION. Throughout this section, X will always denote a pre-Hilbert B-module, and the set of all double B-centralizers will always be denoted by $M(X, B)$.

For a pre-Hilbert B-module X, we let X' denote the set of bounded B-module maps of X into B (i.e. $T(xb)=(Tx)b$, $x\in X$, $b\in B$). Each $x\in X$ gives rise to a map $\hat{x}\in X'$ defined by $\hat{x}(y)=<y, x>$ for $y\in X$.
We will call X *self-dual* if $\hat{X}=X'$. For a trivial example, B is itself a self-dual Hilbert B-module.

PROPOSITION 2.6. *Let $T'' : B\longrightarrow X$ be a bounded module map. Then there is a unique bounded module map $T' : X\longrightarrow B$ such that $(T', T'')\in M(X, B)$.*

PROOF. [1] Proposition 3.4 shows that there is a bounded module map $T' : X\longrightarrow B$ such that $(T', T'')\in M(X, B)$. If there is a bounded module map

$S':X\longrightarrow B$ such that $(S', T'')\in M(X, B)$, then

$$b^*S'x=\langle x, T''b\rangle_X=b^*T'x\ \forall b\in B,\ x\in X.$$

If $b=e$, then $S'=T'$.

PROPOSITION 2.7. *Let* $(T', T'')\in M(X, B)$. *Then* $\|T'\|=\|T''\|$.

PROOF. $\|T''\|=\sup\{\|T''b\|_X : \|b\|\leq 1\}$

$$=\sup\{\|\langle T''b, x\rangle\| : \|b\|\leq 1,\ \|x\|_X\leq 1\}$$
$$=\sup\{\|b^*T'x\| : \|b\|\leq 1,\ \|x\|_X\leq 1\}$$
$$=\sup\{\|T'x\| : \|x\|_X\leq 1\}$$
$$=\|T'\|$$

DEFINITION 2.8. Let $(T', T'')\in M(X, B)$, $(S', S'')\in M(X, B)$, α a complex number. We define a vector space and norm structure on $M(X, B)$ as follows

(i) $(T', T'')+(S', S'')=(T'+S', T''+S'')$

(ii) $\alpha(T', T'')=(\alpha T', \bar{\alpha}T'')$

(iii) $\|(T', T'')\|=\|T'\|$ (or $\|T''\|$)

In [1], the scalar multiplication on X' was defined by $(\alpha T)(x)=\bar{\alpha}T(x)$ for $\alpha\in C$, $T\in X'$. Similarly, we define a scalar multiplication on B'.

PROPOSITION 2.9. *$M(X,B)$ is a pre-Hilbert B-module.*

PROOF. Let $(T', T'')\in M(X, B)$. If we define $(T'\cdot b)(x)=b^*T'(x)$, $(T''\cdot b)(y)=T''(by)$, and $(T', T'')\cdot b=(T'\cdot b, T''\cdot b)$ $(x\in X,\ y,\ b\in B)$, then $\langle x, (T''\cdot b)(y)\rangle=\langle x, T''(by)\rangle=y^*b^*T'(x)=y^*(T'\cdot b)(x)$. Hence, $(T'\cdot b, T''\cdot b)\in M(X, B)$. Therefore $M(X, B)$ is a right B-module. We define $\langle\cdot,\cdot\rangle : M(X, B)\times M(X, B)\longrightarrow B$ by $\langle(T', T''), (S', S'')\rangle=\langle T''(e), S''(e)\rangle_X$. Since $(\alpha T)(x)=\bar{\alpha}T(x)$ ($T\in X'$ or $T\in B'$), $\langle,\rangle$ is a conjugate-bilinear map. It is easy to see that the map $\langle,\rangle$ satisfies (i) and (ii) of the definition of a B-valued inner product. Finally,

$$\begin{aligned}\langle(T', T'')\cdot b, (S', S'')\rangle&=\langle(T'\cdot b, T''\cdot b), (S', S'')\rangle\\&=\langle(T''\cdot b)(e), S''(e)\rangle_X\\&=\langle T''\cdot(e)b, S''(e)\rangle_X\\&=\langle T''(e), S''(e)\rangle_X\, b\\&=\langle(T', T''), (S', S'')\rangle b\end{aligned}$$

Hence $M(X, B)$ is a pre-Hilbert B-module.

THEOREM 2.10. *Let X be a pre-Hilbert B-module. Then X and $M(X, B)$ are isomorphic as pre-Hilbert B-modules.*

PROOF. Define $F'_{(x)}: X \longrightarrow B$ by $F'_{(x)}(y) = \langle y, x \rangle$,

$F''_{(x)}: B \longrightarrow X$ by $F''_{(x)}(b) = xb$, and

$F: X \longrightarrow M(X, B)$ by $F(x) = (F'_{(x)}, F''_{(x)})$

for $x, y \in X$, $b \in B$. Indeed, $\langle y, F''_{(x)}(b) \rangle = \langle y, xb \rangle = b^* \langle y, x \rangle = b^* F'_{(x)}(y)$ for x, y in X and $b \in B$. Therefore $F'_{(x)}$, $F''_{(x)}$, and F are well-defined. Hence we have

$$F'_{(\alpha x)}(y) = \langle y, \alpha x \rangle = \bar{\alpha} F'_{(x)}(y) = (\alpha F'_{(x)})(y)$$
$$F''_{(\alpha x)}(b) = \alpha x b = \alpha F''_{(x)}(b) = (\bar{\alpha} F''_{(x)})(b) \ \forall x, y \in X, \ b \in B, \ \alpha \in C.$$

Hence we know that F is a linear map. Furthermore

$$\begin{aligned} \|F(x)\| = \|F'_{(x)}\| &= \sup\{\|F'_{(x)}(y)\| : y \in X, \ \|y\|_X \leq 1\} \\ &= \sup\{\| \langle y, x \rangle \| : y \in X, \ \|y\|_X \leq 1\} \\ &= \|x\|_X \end{aligned}$$

If $(T', T'') \in M(X, B)$, then $b^* T'(x) = \langle x, T''(b) \rangle_X$ for $x \in X$, $b \in B$. Fix $b = e$, we have $T'(x) = \langle x, T''(e) \rangle_X$. Thus $T' = \widehat{T''(e)}$, by W.L. Paschke [1] Define a mapping $G: M(X, B) \longrightarrow X$ by $G((T', T'')) = T''(e)$. Since $\|T''(e)\| = \|\widehat{T''(e)}\| = \|T'\|$, G is an isometry. It is easy to see that F and G are module maps. In fact, one checks that they are inverses of each other. By [1] Theorem 2.8, applied to the map F and its inverse G, we have $\langle x, x \rangle_X = \langle F(x), F(x) \rangle_{M(X,B)}$. It is not hard to show that X and $M(X, B)$ have the same inner product.

COROLLARY 2.11. *Let X be a pre-Hilbert B-module. Then X is a self-dual if and only if every map T in X' has a bounded module map T^* such that $(T, T^*) \in M(X, B)$.*

COROLLARY 2.12. *If X is a self-dual Hilbert B-module, then $n(X') = n(B')$.* (We denote that $n(A)$ is the number of A)

PROOF. Proposition 2.6 and Corollary 2.11.

COROLLARY 2.13. *Let B be a B^*-algebra. Then $B = \{T(e) \mid T \in \mathscr{B}(B)\}$, where $\mathscr{B}(B)$ is the set of all bounded module maps on B.*

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