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ON STABILITY OF PERTURBED THIRD ORDER LINEAR DIFFERENTIAL EQUATION

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The equation considered here is of the form

$$\dddot{u}+a\ddot{u}+\dot{u}+au=\mu h\ (u,\ \dot{u},\ \ddot{u},\ \mu) \tag{1}$$

in which a is a constant, h is a real analytic function in all its arguments and μ is a real small parameter. The dots indicate differentiation with respect to the time t.

Any equation in the form

$$\dddot{u}+a\ddot{u}+b\dot{u}+cu=\mu h\ (u,\ \dot{u},\ \ddot{u},\ \mu)$$

where a, b and c are constants reduced by certain transformation to equation (1).

The main assumption in this paper is that the unperturbed equation

$$\dddot{u}+a\ddot{u}+\dot{u}+au=0$$

has a pair of imaginary eigenvalues. This problem is quite different from that considered by Ezeilo [2].

Let

$$z_1=u,\ \ z_2=\dot{u},\ \ z_3=\ddot{u}$$

then equation (1) is reduced to the system

$$\dot{z}=Bz+\mu g\ (z,\ \mu) \tag{2}$$

where

$$z=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},\quad B=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -1 & -a \end{bmatrix},\quad g(z,\ \mu)=\begin{bmatrix} 0 \\ 0 \\ h(z_1,\ z_2,\ z_3,\ \mu) \end{bmatrix}$$

The transformation $z=QX$ with $\det Q\neq 0$ will let equation (2) takes the form

$$\dot{x}=Ax+\mu\ f(x,\ u) \tag{3}$$

where

$$x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\quad A=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad Q=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -a \\ -1 & 0 & a^2 \end{bmatrix}$$

and $f=Q^{-1}g$.

It is easy to see that the eigenvalues of A are $-a$, $+i$, $-i$, i.e. A has one real and two pure imaginary roots which are in accordance with our assumption.

Through this paper the following abbreviations will be used:

$$\begin{aligned} R(x,\mu) &= h(Qx,\mu) \\ r(t) &= R(b\cos t,\ b\sin t,\ 0\,;0) \\ r_i(t) &= R'_{x_i}(b\cos t,\ b\sin t,\ 0\,;0) \\ r_\mu(t) &= R'_\mu(b\cos t,\ b\sin t,\ 0\,;0) \end{aligned}$$

where b is a real constant.

THEOREM 1. *If a real constant $b\neq 0$ exist for which*

$$\int_0^{2\pi} r(-t)[\cos t - a\sin t]\,dt \equiv F(b)=0$$

and

$$\frac{\partial F(b)}{\partial b}=\int_0^{2\pi}[\cos t - a\sin t]\,[r_1(-t)\cos t - r_2(-t)\sin t]\,dt\neq 0$$ *then for $|\mu|$ sufficiently small equation* (1) *has a unique nonconstant periodic solution $p(t,\mu)$ of period $T(\mu)=2+\delta(\mu)$ such that:*

$$p(t,0)=b\cos t,$$

$$T(0)=2\pi,\ \textit{and}$$

$$\lim_{\mu\to 0}\frac{\delta(\mu)}{\mu}=c=\frac{1}{b(a^2+1)}\int_0^{2\pi} r(-t)\,[\sin t + a\cos t]\,dt.$$

The solution $p(t;\mu)$ is analytic for all t and $|\mu|$ sufficiently small.

The proof of this theorem is analogous to that of Theorems 4.1 and 4.2 Chapter 14, [1], and therefore be omitted.

Let $q(t;\mu)$ be the periodic solution of (3). The first variational system of system (3) corresponding to $q(t,\mu)$ is

$$y=[A'_x+\mu f'_x(q(t;\mu);\mu)]\,y \tag{4}$$

where $y=[y_1, y_2, y_3]^T$, $A'_x=\dfrac{\partial A}{\partial x}$.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if $|\mu|$ is sufficiently small, $\mu\neq 0$ and*

$$\mu \int_0^{2\pi} [r_1(t)+ar_2(t)]\,dt>0 \tag{5}$$

then the characteristic multipliers of (4) *satisfying the following conditions*

$$\lambda_1(\mu)=1, \quad |\lambda_2(\mu)|<1 \text{ and } |\lambda_3(\mu)|<1,$$

thus the periodic solution $p(t, \mu)$ *of* (1) *is orbitally asymptotically stable.*

PROOF. By Liouville's formula we have

$$\lambda_1\cdot\lambda_2\cdot\lambda_3=\exp\left\{-aT-\frac{\mu}{1+a^2}\int_0^{T(\mu)}[r_1+ar_2-r_3]\ dt\right\}$$

Since $\dot{q}(t, \mu)$ is a periodic solution of (4), then one of the characteristic multipliers of system (4) is in modulus equal to 1, say λ_1, i.e. $|\lambda_1(\mu)|=1$. Thus

$$\lambda_2(\mu).\ \lambda_3(\mu)=\exp\left\{-aT-\frac{\mu}{1+a^2}\int_0^{T(\mu)}[r_1+ar_2-r_3]\ dt\right\}$$

where $\lambda_3(\mu)$ is the characteristic multiplier of (4) for which $\lambda_3(0)=e^{-2\pi a}$. Thus the condition $|\lambda_3(\mu)|=e^{-Ta}<1$ holds if $|\mu|$ is sufficiently small.

The relation between the characteristic multiplier and the corresponding characteristic exponent is given by

$$\lambda=e^{T\rho} \tag{7}$$

It is obviously $\lambda_2(\mu)$, r_i, q and T are analytic functions of μ for $|\mu|$ sufficiently small. Hence:

$$|\lambda_2(\mu)|=\exp\left\{\frac{-\mu}{1+a^2}\int_0^{2\pi}[r_1+ar_2-r_3]\,dt-\mu 2\pi\rho'_2(0)+0(u)\right\} \tag{8}$$

where $\rho'_2(0)=[d\rho_2/d\mu]_\mu=0$. On the other hand from (7) we have

$$\rho_2(0)=\frac{1}{2\pi}(\lambda'_2(0)e^{2\pi a}+ac). \tag{9}$$

Using Loud Theorem 1, [4], or Farkas method [3] one can determine $\lambda'_2(0)$ as follows

$$\lambda'_2(0)=-ace^{-2\pi a}+\frac{1}{1+a^2}\ e^{-2\pi a}\int_0^{2\pi} m_3(t)\ dt. \tag{10}$$

Substitution (10) into (9) the condition $|\lambda_2(\mu)|<1$ is satisfied. This completes the proof of the theorem.

REMARKS. (1) If

$$\mu\int_0^{2\pi}[r_1(t)+ar_2(t)]\,dt<0$$

The conditions of the Theorem 2 do not hold and thus the periodic solution $p(t,\ \mu)$ of (1) is unstable.

(2) If

$$\int_0^{2\pi}[r_1(t)+ar_2(t)]\,dt=0$$

the stability condition depends on the coefficient of the second approximation of μ^2 in (8).

(3) The above method can be extended to the real perturbed linear equation of order n which takes the following form

$$u^{(n)}+a_1u^{(n-1)}+a_2u^{(n-2)}+\dots a\,\acute{u}+a\,u=\mu h(u,\ u,\ u\dots,\ u^{(n-1)},\ \mu)$$

where the characteristic equation

$$r^n+a_1r^{n-1}+\dots a_{n-1}r+a_n=0$$

of the corresponding unperturbed equation has two pure imaginary roots and all the other roots are simple and have negative real parts.

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REFERENCES

[1] E. Coddington & N. Levinson, *Theory of ordinary differential equations*, Mcgraw-Hill, New York (1955).

[2] J. Ezeilo, *Further results for the solutions of a third order differential equation*, Proc. Camb. Phil. Soc. (1963), 59, 111—116.

[3] M. Farkas, *Determination of controllably periodic perturbed solutions by Poincare's method*, Studia Sci. Math. Hungar. (1972) 259—268.

[4] W. Loud, *Periodic Solutions of a perturbed autonomous system*, Annals of Math. 70 (1959), 496—529.