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ON LINEAR SYSTEMS

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1. Consider the linear homogeneous system

$$x'=A(t)x,\ \ '=\frac{d}{dt} \tag{1}$$

where x is n- dimensional vector, $A(t)$ is $n\times n$ matrix of complex continuous function such that $A(t+\omega)=A(t)$, $-\infty<t<\infty$ for some constant $\omega>0$.

The fundamental result for such systems concerns the full representation of the fundamental matrix solution $\Phi(t)$ of system (1) as:

(i) $\Phi(t+w)=\Phi(t)C$

where C is $n\times n$ nonsingular constant matrix and $\Phi(0)=U$, the unit matrix.

(ii) $\Phi(t)=p(t)e^{Rt}$

where $p(t+w)=p(t)$, and $p(t)$ is a nonsingular matrix, R is $n\times n$ constant matrix defined by $e^{Rw}=C=\Phi(w)$. System (1) is known as Floquet system (F.S.).

The following abbreviation

$$[S,\ T]=ST-TS$$
$$B(t,\ w)=A(t+w)-A(t)$$

will be used throughout this paper.

Let $\Psi(t,\ w)$ denote the fundamental matrix solution of the system

$$y'=B(t,\ w)y \tag{2}$$

for which $\Psi(0,\ U)=U$ (the unit matrix) holds.

DEFINITION. The system (1) is said to be a *generalized Floquet system*, (G.F.S.) if and only if

$$[B(t,\ w),\ \Phi]=0,\ \ -\infty<t<\infty.$$

LEMMA 1. *If* $[B(S,\ w),\ A(t)]=0$, *then the system* (1) *is a G.F.S.*

PROOF. Differentiating $[B(S,\ w),\ \Phi(t)]$ with respect to t, we have

$$[B(S,\ w),\ \Phi(t)]'=B\Phi'-\Phi'B$$
$$=BA\Phi-A\Phi B.$$

Since $[B(S,\ w),\ A(t)]=0$, so

$$[B(S,\ w)\ \Phi(t)]'=A(t)\ [B(S,\ w),\ \Phi(t)].$$

Thus $[B(S,\ w),\ \Phi(t)]$ satisfies a linear matrix differential system with conditions that at $t=0$, we have $[B(S,\ w),\ \Phi(0)]=0$ and consequently $[B(S,\ w),\ \Phi(t)]=0$; therefore in particular, $[B(t,\ w),\ \Phi(t)]=0$ for all t. This completes the proof.

LEMMA 2. *If the system* (1) *is a G.F.S., then*:

(i) $[\Phi(w),\ \Psi(t,\ w)]=0$, *provided* $[\Phi(w),\ B(t,\ w)]=0$

(ii) $[R,\ \Psi(t,\ w)]=0$, *provided* $[R,\ B(t,\ w)]=0$

(iii) $[\Psi(t,\ w),\ e^{-Rt}]=0$, *provided* $[R,\ B(t,\ w)]=0$

(iv) $[R,\ e^{-Rt}]=0$.

PROOF. (i) Differentiating $[\Phi(w),\ \Psi(t,\ w)]$ with respect to t, we have

$$[\Phi(w),\ \Psi(t,\ w)]'=\Phi\Psi'-\Psi'\Phi$$
$$=\Phi B\Psi-B\Psi\Phi$$

But since $[\Phi(w),\ B(t,\ w)]=0$, so $[\Phi(w),\ \Psi(t,\ w)]'=B[\Phi(w),\ \Psi(t,\ w)]$. Thus $[\Phi(w),\ \Psi(t,\ w)]$ satisfies a linear matrix differential system with conditions that at $t=0$ we have $[\Phi(w),\ \Psi(0,\ w)]=0$ and hence $[\Phi(w),\ \Psi(t,\ w)]=0$ for all t. This completes the proof of part (i). For parts (ii) and (iii), we follow the same technique. The proof of past (iv) follows from the definition.

THEOREM 1. *Let the system* (1) *be a G.F.S. and* Φ, Ψ *be the fundamental matrix solutions for the system* (1) & (2) *respectively, then*

$$\text{(i)}\quad \Psi(t,\ w)=\Phi^{-1}(t)\Phi(t+w)\Phi^{-1}(w) \tag{3}$$

$$\text{(ii)}\quad \Phi(t+nw)=\Phi(t)\,[\Phi(w)]^{n}\prod_{r=0}^{n-1}\Psi(t+rw,\ w) \tag{4}$$

$$\text{(iii)}\quad P(t+nw)=P(t)\prod_{r=0}^{n-1}\Psi(t+rw,\ w) \tag{5}$$

provided $[R,\ B(t,\ w)]=0$.

PROOF. Relation (3) can be written as

$$\Psi(t,\ w)=Z(t,\ w)\Phi^{-1}(w),$$

where $$Z(t,\ w)=\Phi^{-1}(t)\Phi(t+w) \tag{6}$$

Differentiating $Z(t,\ w)$ with respect to t, we obtain

$$Z'(t,\ w)=(\Phi^{-1})'\Phi(t+w)+\Phi^{-1}(t)\Phi'(t+w)$$
$$=\Phi^{-1}(t)\,[-A(t)+A(t+w)]\,\Phi(t+w)$$

since $(\Phi^{-1})=-\Phi^{-1}(t)A(t)$. Thus we have

$$Z'(t,\ w)=\Phi^{-1}(t)B(t,\ w)\Phi(t+w).$$

Using (6), we obtain

$$Z'(t,\ w)=\Phi^{-1}(t)\ B(t,\ w)\Phi(t)Z(t,\ w)$$
$$=B(t,\ w)Z(t,\ w),$$

since

$$[\Phi(t),\ B(t,\ w)]=0$$

by assumption. Thus $Z(t,\ w)$ is a solution of system (2) such that $Z(0,\ w)=\Phi(w)$. Relation (3) follows directly from the definition of $Z(t,\ w)$.

(ii) Rewrite relation (3) in the form

$$\Phi(t+w)=\Phi(t)\Psi(t,\ w)\ \Phi(w) \tag{7}$$

then using Lemma 2-i, equation (7) takes the forms

$$\Phi(t+w)=\Phi(t)\Phi(w)\ \Psi(t,\ w) \tag{8}$$

Thus relation (4) is true for $n=1$. Replacing t by $t+w$ in (8), we have

$$\Phi(t+2w)=\Phi(t+w)\Phi(w)\Psi(t+w,\ w).$$

Using (8) and Lemma 2-i, we obtain

$$\Phi(t+2w)=\Phi(t)\,[\Phi(w)]^2\ \Psi(t,\ w)\ \Psi(t+w,\ w)$$
$$=\Phi(t)\,[\Phi(w)]^2\prod_{r=0}^{1}(t+rw,\ w).$$

Thus relation (4) is true for $n=2$. Now the relation (4) is true for $n=1,2$. Next, we employ the principle of induction. Let relation (4) be true for $n=K$, i.e. we have

$$\Phi(t+Kw)=\Phi(t)\quad[\Phi(w)]^k\prod_{r=0}^{k-1}\Psi(t+rw,\ w)$$

Replacing t by $t+w$, we have

$$\Phi(t+(K+1)w)=\Phi(t+w)\quad[\Phi(w)]^k\prod_{r=0}^{k-1}(t+(r+1)w,\ w)$$

Using (7) and Lemma 2-i, we obtain

$$\Phi(t+(K+1)w)=\Phi(t)\,[\Phi(w)]^{K+1}\Psi(t,\ w)\prod_{r=0}^{K-1}\Psi(t+(r+1)w,\ w)$$
$$=\Phi(t)\,[\Phi(w)]^{K+1}\prod_{r=0}^{K}\Psi(t+rw,\ w)$$

Hence relation (4) is true for $K+1$. Thus it is true for all values of n. This completes the proof of (ii).

To prove (5) we note that $P(t)=\Phi(t)e^{-Rt}$, therefore by replacing t by $t+w$, we obtain

$$p(t+w)=\Phi(t+w)e^{-R(t+w)}$$

By using (3), we get

$$p(t+w)=\Phi(t)\Psi(t,\ w)\ \Phi(w)e^{-Rt}e^{-Rw}$$
$$=\Phi(t)\Psi(t,\ w)e^{-Rt}$$

since $\Phi(w)=e^{Rw}$. Using Lemma 2-iii, we obtain

$$p(t+w)=\Phi(t)e^{-Rt}\Psi(t,\ w)$$
$$=p(t)\Psi(t,\ w)$$

Thus relation (5) is true for $n=1$. Replacing to by $t+w$, we get

$$p(t+2w)=p(t+w)\ (t+w,\ w)$$
$$=p(t)\Psi(t,\ w)\ (t+w,\ w)$$
$$=p(t)\prod_{r=0}^{1}(t+rw,\ w)$$

Thus relation (5) is true for $n=2$. Using the technique of induction as for relation (4), we can see that the relation (5) is true for all n. This completes the proof of the theorem.

We shall consider the case in which $B(t,w)=B_1$, where B_1 is a constant. It is clear that the system (1) is a G.F.S if $[B_1,\ A(t)]=0$ (by Lemma 1). The fundamental matrix solution $\Psi(t,\ w)$ of system (2) takes the form

$$\Psi(t,\ w)=e^{B_1 t}.$$

The relations (3), (4) & (5) reduce to

$$\Phi(t+w)=\Phi(t)e^{B_1 t}\Phi(w) \tag{9}$$

$$\Phi(t+nw)=\Phi(t)\,[\Phi(w)]^n e^{B_1\left[nt+\frac{1}{2}n(n-1)\right]}, \tag{10}$$

and

$$p(t+nw)=p(t)e^{B_1\left[nt+\frac{1}{2}n(n-1)\right]} \tag{11}$$

respectively.

Expression (9) enables us to study the stability criteria and the following result may be obtained.

THEOREM 2. *If the characteristic roots of* B_1 *have negative (positive) real parts, then the trivial solution of* (1) *is asymptotically stable (unstable).*

PROOF. The proof of this theorem is an immediate consequence of Theorems 1.1 & 1.2 of Chapter 13 in Coddington & Levinson [1].

EXAMPLE. Let

$$A(t)=\begin{bmatrix} at+p(t) & b \\ c & at+q(t) \end{bmatrix},$$

where $p(t)$ and $q(t)$ are periodic functions with least period w and a, b and c

are constants. Then

$$B(t, w)=A(t+w)-A(t)=awU=B_1,$$

where U is the unit matrix.

Since $w>0$ (by assumption), it is clear that if $Re\ a<0$ then the zero solution of system (1) is assumptotically stable and if $Re\ a>0$, then the zero solution is unstable. If $a=0$, then $B_1=0$ & consequently $A(t+w)=A(t)$, i.e. the system reduces to the Floquet system.

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