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DECOMPOSITION OF RECURRENT CURVATURE TENSOR FIELD IN 2-R-GENERALISED FINSLER SPACES

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0. Introduction

Author and Sinha [2] have defined generalised Finsler spaces of first order. Recently author has defined generalised Finsler spaces of second order also and denoted them by $2\text{-}RG\text{-}\bar{F}_n$ and $2\text{-}RG\text{-}F_n$. The object of present paper is to decompose the 2-recurrent curvature tensor fields in these spaces and also to study the important properties of decomposition tensor fields.

We consider n-dimensional generalised Finsler spaces GF_n in which connection parameters for the locally Minkowskian and locally Euclidean spaces are denoted by P^{*i}_{jk} and Γ^{*i}_{jk} respectively. Let T^i be a vector field of GF_n then the two pro cesses of di ferentiation are defined as under

$$T^i_{,j}=\partial_j T^i+\partial_j \dot{x}^h \dot{\partial}_h T^i+P^{*i}_{kj}T^k \quad {}^{2)} \tag{0.1}$$

and

$$T^i|_j=\partial_j T^i-\Gamma^h_{kj}\dot{\partial}_h T^i \dot{x}^k+\Gamma^{*i}_{kj}T^k \tag{0.2}$$

where

$$\Gamma^i_{jk}=\Gamma^{*i}_{jk}+C^i_{jh}\Gamma^{*h}_{rk}\dot{x}^r \tag{0.3}$$

and

$$C_{ijk}=\frac{1}{4}\dot{\partial}^3_{ijk}F^2(x, \dot{x}). \tag{0.4}$$

With the help of above covariant differentiations two curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} are defined.

The commutation formula involving the curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} are given as follows [1]:

$$2T^i_{,[jk]}=T^h\bar{K}^i_{hkj}-2T^i_{,h}\Delta^h_{[jk]} \quad {}^{3)} \tag{0.5}$$

1) Numbers in the brackets refer to the references.
2) $\partial_j=\partial/\partial x^j$ and $\dot{\partial}_j=\partial/\partial \dot{x}^j$.
3) $2T^i_{,[jk]}=T^i_{,jk}-T^i_{,kj}$.

and

$$2T^i|_{[jk]}=\dot{\partial}_h T^i K^h_{ojk}+T^h K^i_{hkj}-2T^i|_h \Delta^h_{[jk]}, \tag{0.6}$$

where

$$\Gamma^{*i}_{[jk]}=P^{*i}_{[jk]}=\Delta^i_{[jk]} \tag{0.7}$$

and

$$K^i_{okh}=K^i_{jkh}l^j \tag{0.8}$$

We also have

$$\dot{\partial}_l \Gamma^{*i}_{jk}\dot{x}^j\dot{x}^k=0 \tag{0.8 a}$$

In GF_n, the curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} satisfy the following identities

$$\bar{K}^i_{jkh}=-\bar{K}^i_{jkh}, \tag{0.9}$$

$$K^i_{jkh}=-K^i_{jhk},\ K^i_{jki}=K_{jk}, \tag{0.10}$$

$$\bar{K}^i_{jkh}+\bar{K}^i_{khj}+\bar{K}^i_{hjk}=2\Delta_{[j|k|h];l}g^{il}, \tag{0.11}$$

where (;) denotes covariant derivative based upon the connection parameter given by $Q^*_{jkh}=P^*_{jkh}+g_{(jk),h}$

$$K^i_{jkh}+K^i_{khj}+K^i_{hjk}=2\Delta_{[j|k|h],l}g^{il}, \tag{0.12}$$

where (,) denotes covariant derivative based upon the connection parameter given by $R^*_{jkh}=\Gamma^*_{jkh}$

$$\bar{K}^i_{jkh,l}+\bar{K}^i_{jhl,k}+\bar{K}^i_{jlk,h}+2[\bar{K}^i_{jmk}P^{*m}_{[lh]}+\bar{K}^i_{jmh}P^{*m}_{[kl]}+\bar{K}^i_{jml}P^{*m}_{[hk]}]=0 \tag{0.13}$$

and

$$\begin{aligned}K^i_{jkh|l}+K^i_{jhl|k}+K^i_{jlk|h}+F[K^m_{ohk}\dot{\partial}_m\Gamma^{*i}_{jl}+K^m_{olh}\dot{\partial}_m\Gamma^{*i}_{jk}+K^m_{okl}\dot{\partial}_m\Gamma^{*i}_{hj}]\\ =2[K^i_{jml}\Delta^m_{[kh]}+K^i_{jmk}\Delta^m_{[hl]}+K^i_{jmh}\Delta^m_{[lk]}].\end{aligned} \tag{0.14}$$

Sinha and Singh [2] have defined recurrent curvature tensor fields in GF_n as under:

The GF_n in which there exists a non-zero vector v_l such that the curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} satisfy the relations

$$\bar{K}^i_{jkh,l}=v_l\bar{K}^i_{jkh} \tag{0.15}$$

and

(0.16) $$K^i_{jkh|l}=v_l K^i_{jkh}$$

respectively, are said to be recurrent GF_n (or in brief RGF_n) and the curvature tensor fields of these spaces are called recurrent curvature tensor fields. Here v_l is known as recurrence vector field.

Contracting the indices i and h in (0.16) we find

(0.17) $$K_{jk|l}=v_l K_{jk}$$

Transvecting (0.16) by l^j and using (0.8), we write

(0.18) $$K^i_{okh|l}=v_l K^i_{okh}.$$

Author [3] has defined recurrent generalised Finsler spaces of second order and denoted them by $2\text{-}RG\bar{F}_n$ and $2\text{-}RGF_n$ as under:

The n-dimensional generalised Finsler spaces GF_n, in which relative and Cartan curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} satisfy the relations

(0.19) $$\bar{K}^i_{jkh,lm}=a_{lm}\bar{K}^i_{jkh},\ \ \bar{K}^i_{jkh}\neq 0$$

and

(0.20) $$K^i_{jkh|lm}=a_{lm}K^i_{jkh},\ \ K^i_{jkh}\neq 0$$

respectively, where a_{lm} is a non-zero recurrence tensor field, are defined as recurrent generalised Finsler spaces. Also the curvature tensor fields which satisfy (0.19) and (0.20) are defined as recurrent curvature tensor fields of second order.

Contracting the indices i and h in (0.20), we find

(0.21) $$K_{jk|lm}=a_{lm}K_{jk}$$

along with $K_{jk}\neq 0$.

Transvecting (0.20) by l^j we obtain

(0.22) $$K^i_{okh|lm}=a_{lm}K^i_{okh}$$

in view of (0.8).

Author [3] has proved the following theorem which is of further use.

THEOREM. *The recurrent generalised Finsler spaces of first order for which the recurrence vector field v_l satisfies*

(0.23) $$a_{lm}=v_{l,m}+v_l v_m\neq 0$$

and

$$a_{lm}=v_{l|m}+v_l v_m \neq 0 \tag{0.24}$$

are also recurrent generalised Finsler spaces of second order but the converse is not true in general.

1. Decomposition of curvature tensor field $\tilde{K}^i_{jkh}$ in 2-$RG\tilde{F}_n$

We consider the decomposition of relative curvature tensor field $\tilde{K}^i_{jkh}$ in the following manner

$$\tilde{K}^i_{jkh}=X^i\,\phi_{jkh}, \tag{1.1}$$

where ϕ_{jkh} is a non-zero decomposition tensor field and X^i is any vector field such that

$$X^i\,v_i=1. \tag{1.2}$$

Furthermore, we decompose the tensor field ϕ_{jkh} as under

$$\phi_{jkh}=v_j\,\phi_{kh} \tag{1.3}$$

According to theorem stated in section 0, every recurrent generalised Finsler space of first is also recurrent generalised Finsler space of second order but the converse is not true, in general. Thus the theorems stated in [4] must also hold good in this space and hence we can directly state the following theorems for the above decomposition in 2-$RG\tilde{F}_n$.

THEOREM 1.1. *In 2-$RG\tilde{F}_n$, the decomposition tensor fields ϕ_{jkh} and ϕ_{kh} satisfy the following identities*

(1.4) (a) $$\phi_{jkh}=-\phi_{jhk}.$$

(b) $$\phi_{kh}=-\phi_{hk},$$

(c) $$\phi_{jkh}+\phi_{khj}+\phi_{hjk}=2\Delta_{[j|k|h];l}\,v^l$$

and

(d) $$\phi_{km}\overset{*}{P}{}^m_{[lh]}+\phi_{hm}\overset{*}{P}{}^m_{[kl]}+\phi_{lm}\overset{*}{P}{}^m_{[hk]}=\Delta_{[l|k|h];m}\,v^m.$$

THEOREM 1.2. *In 2-$RG\tilde{F}_n$, the necessary and sufficient condition for the decomposition tensor fields ϕ_{jkh} and ϕ_{kh} to be recurrent*

(1.5) (a) $$\phi_{jkh,l}=v_l\,\phi_{jkh}$$

(b) $$\phi_{kh,l}=v_l\,\phi_{kh}$$

is that the vector field X^i is covariant constant.

THEOREM 1.3. *In 2-$RG\tilde{F}_n$, the decomposition tensor fields ϕ_{jkh} and ϕ_{kh} satisfy the Bianchi identities*

(1.6) (a) $\phi_{jkh,l}+\phi_{jhl,k}+\phi_{jlk,h}=2[P^{*m}_{[lh]}\phi_{jkm}+P^{*m}_{[kl]}\phi_{jhm}+P^{*m}_{[hk]}\phi_{jlm}]$

and

(b) $\phi_{kh,l}+\phi_{hl,k}+\phi_{lk,h}=2[P^{*m}_{[lh]}\phi_{km}+P^{*m}_{[kl]}\phi_{hm}+P^{*m}_{[hk]}\phi_{lm}]$

respectively along with the condition that the vector field X^i is covariant constant.

Differentiating (1.5a) covariantly and using (1.5a) we find

(1.7) $$\phi_{jkh,lm}=(v_{l,m}+v_l v_m)\phi_{jkh}.$$

Noting (1.23) in the equation (1.7), it gives

(1.8) $$\phi_{jkh,lm}=a_{lm}\phi_{jkh}.$$

In view of (1.2) and (1.3), the equation (1.8) yields

(1.9) $$\phi_{kh,lm}=a_{lm}\phi_{kh}.$$

Hence we state

THEOREM 1.4. *In 2-$RG\tilde{F}_n$, if the decomposition tensor fields ϕ_{jkh} and ϕ_{kh} are first order recurrent, then these tensor fields are also second order recurrent but the converse is not true.*

Commuting the indices l and m in (0.19) and using (0.5) we obtain

(1.10) $\bar{K}^r_{jkh}\bar{K}^i_{rml}-\bar{K}^i_{rkh}\bar{K}^r_{jml}-\bar{K}^i_{jrh}\bar{K}^r_{kml}-\bar{K}^i_{jkr}\bar{K}^r_{hml}-2\bar{K}^i_{jkh,r}A^r_{[lm]}=2a_{[lm]}\bar{K}^i_{jkh}$

In view of (1.1), (1.2) and (1.3) it becomes

(1.11) $-X^iX^r v_j v_k\phi_{rh}\phi_{ml}-X^iX^r v_j v_h\phi_{kr}\phi_{ml}-2X^i v_j\phi_{kh,r}\Delta^r_{[lm]}=2a_{[lm]}X^i v_j\phi_{kh}.$

Transvecting (1.11) by v_i X^j and making use of (1.2), (1.3) and (1.4a), we have

(1.12) $$X^r\phi_{khr}-X^r\phi_{hkr}\phi_{ml}-2\phi_{kh,r}\Delta^r_{[lm]}=2a_{[lm]}\phi_{kh}$$

Here we consider $\phi_{khr}X^r=0$, the equation (1.12) becomes

(1.13) $$v_r\phi_{kh}\Delta^r_{[lm]}=a_{[ml]}\phi_{kh}$$

by means of (1.5b).

Since $\phi_{kh}\neq 0$, the equations (1.13) yields

(1.14) $$v_r\Delta^r_{[lm]}=a_{[lm]}\,.$$

Conversely if (1.14) is true the equation (1.11) takes the form

(1.15) $$X^iX^rv_jv_k\phi_{rh}\phi_{ml}+X^iX^rv_jv_h\phi_{kr}\phi_{ml}=0.$$

Multiplying (1.15) by v_iX^j and using (1.2), (1.3) and (1.4a) we get

(1.16) $$X^r\phi_{hkr}\phi_{ml}-X^r\phi_{khr}\phi_{ml}=0.$$

Since $\phi_{ml}\neq 0$, therefore (1.16) gives

(1.17) $$X^r\phi_{hkr}=X^r\phi_{khr}$$

which implies

(1.18) $$\phi_{hkr}X^r=0.$$

Thus we state.

THEOREM 1.5. *In 2-$RG\tilde{F}_n$ the necessary and sufficient condition for the relation*

$$v_r\Delta^r_{[lm]}=a_{[ml]}$$

to be true is that

$$\phi_{hkr}X^r=0.$$

Simplifying (1.4c) by means of (1.3), we find

(1.19) $$v_j\phi_{kh}+v_k\phi_{hj}+v_h\phi_{jk}=2\Delta_{[j|k|h];l}v^l\,.$$

Multiplying (1.19) by X^j and using (1.2) and (1.4b), we obtain

(1.20) $$\phi_{kh}+v_k\phi_{hj}X^j-v_h\phi_{kj}X^j=2\Delta_{[j|k|h];l}v^lX^j\,.$$

Let us assume that $\phi_{hj}X^j=0$, then (1.20) takes the form

(1.21) $$\phi_{kh}=2\Delta_{[j|k|h];l}v^lX^j\,.$$

Conversely if, (1.21) is true, the equation (1.19) becomes

(1.22) $$v_k\phi_{jh}=v_h\phi_{jk}$$

in view of (1.4b).

Transvecting (1.22) by X^kX^h and noting (1.2), we have

(1.23) $$\phi_{jh}X^h=\phi_{jk}X^h$$

which implies

(1.24) $$\phi_{jh} X^h = 0.$$

Accordingly we have

THEOREM 1.6. *In 2-$RG\tilde{F}_n$, the necessary and sufficient condition for the decomposition tensor field ϕ_{kh} to be expressed as*

$$\phi_{kh} = 2\Delta_{[j|k|h];l} v^l X^j$$

is that

$$\phi_{hj} X^j = 0.$$

Differentiating (0.13) and simplifying with help of (0.13), (0.15), (0.19), (1.1) and (1.2), we obtain (1.25)

(1.25) $$[\phi_{jkh} a_{ln} + \phi_{jhl} a_{kn} + \phi_{jlk} a_{hn}] - v_n [v_l \phi_{jkh} + v_k \phi_{jhl} + v_h \phi_{jlk}] + 2[\phi_{jmk} \overset{*}{P}{}^m_{[lh],n} + \phi_{jmh} \overset{*}{P}{}^m_{[kl],n} + \phi_{jml} \overset{*}{P}{}^m_{[hk],n}].$$

By means of (0.23), the equation (1.25) reduces to

(1.26) $$v_{l,n} \phi_{jkh} + v_{k,n} \phi_{jhl} + v_{h,n} \phi_{jlk} + 2[\phi_{jmk} \overset{*}{P}{}^m_{[lh],n} + \phi_{jmh} \overset{*}{P}{}^m_{[kl],n} + \phi_{jml} \overset{*}{P}{}^m_{[hk],n}] = 0.$$

Transvecting (1.26) by X^j and noting (1.2) and (1.3), we get

(1.27) $$v_{l,n} \phi_{kh} + v_{k,n} \phi_{hl} + v_{h,n} \phi_{lk} + 2[\phi_{mk} \overset{*}{P}{}^m_{[lh],n} + \phi_{mh} \overset{*}{P}{}^m_{[kl],n} + \phi_{ml} \overset{*}{P}{}^m_{[hk],n}] = 0.$$

Accordingly we have

THEOREM 1.7. *In 2-$RG\tilde{F}_n$, the decomposition tensor fields ϕ_{jkh} and ϕ_{kh} satisfy the following relations*

$$v_{l,n} \phi_{jkh} + v_{k,n} \phi_{jhl} + v_{h,n} \phi_{jlk} + 2[\phi_{jmk} \overset{*}{P}{}^m_{[lh],n} + \phi_{jmh} \overset{*}{P}{}^m_{[kl]} + \phi_{jml} \overset{*}{P}{}^m_{[hk],n}] = 0$$

and

$$v_{l,n} \phi_{kh} + v_{k,n} \phi_{hl} + v_{h,n} \phi_{lk} + 2[\phi_{mk} \overset{*}{P}{}^m_{[lh],n} + \phi_{mh} \overset{*}{P}{}^m_{[kl],n} + \phi_{ml} \overset{*}{P}{}^m_{[hk],n}] = 0$$

respectively.

REMARK 1.1. In 2-$RG\tilde{F}_n$ if the vector field X^i is covariant constant it implies that v_i is also covariant constant.

By virtue of Remark 1.1 we state

COROLLARY 1.1. *In 2-$RG\tilde{F}_n$ the decomposition tensor fields ϕ_{jkh} and ϕ_{kh} satisfy the following relations*

$$\phi_{jmk}P^{*m}_{[lh],n}+\phi_{jmh}P^{*m}_{[kl],n}+\phi_{jml}P^{*m}_{[hk],n}=0$$

and

$$\phi_{mk}P^{*m}_{[lh],n}+\phi_{mh}P^{*m}_{[kl],n}+\phi_{ml}P^{*m}_{[hk],n}=0$$

respectively.

2. Decomposition of curvature tensor field K^i_{jkh} in 2-RGF_n.

In this section we decompose the Cartan's Curvature tensor field K^i_{jkh} in the form

(2.1) $$K^i_{jkh}=X^i\bar{\phi}_{jkh},$$

where $\bar{\phi}_{jkh}$ is a non-zero decomposition tensor field and the vector field X^i satisfies the relation $X^iv_i=1$.

Contracting the indices i and h in (2.1) and using (0.10), we get

(2.2) $$K_{jk}=\bar{\phi}_{jko},$$

where

(2.3) $$\bar{\phi}_{jko}=\bar{\phi}_{jkh}X^h.$$

Transvecting (2.1) by l^j and noting (0.8), we fin.

(2.4) $$\bar{\phi}_{jkh}l^j=\bar{\phi}_{okh},$$

where

(2.5) $$K^i_{okh}=X^i\bar{\phi}_{okh}.$$

Similar to that of Section 2.1, every RGF_n is also 2-RGF_n and hence we can state the following theorems similar to that of RGF_n [4] which are true in 2-RGF_n.

THEOREM 2.1. *In 2-RGF_n, the decomposition tensor field $\bar{\phi}_{jkh}$ and $\bar{\phi}_{okh}$ satisfy the following identities*

(2.6) (a) $$\bar{\phi}_{jkh}=-\bar{\phi}_{jhk}$$

(b) $$\bar{\phi}_{okh}=-\bar{\phi}_{ohk}$$

(c) $$\bar{\phi}_{jkh}+\bar{\phi}_{khj}+\bar{\phi}_{hjk}=2\Delta_{[j|k|h],l}v^l.$$

THEOREM 2.2. *In 2-RGF_n the necessary and sufficient condition for the decomposition tensor fields $\bar{\phi}_{jkh}$* and *$\bar{\phi}_{okh}$ to be recurrent i.e.*

$$(2.7)\ \text{(a)} \qquad \bar{\phi}_{jkh|l}=v_l\bar{\phi}_{jkh}$$
$$\text{(b)} \qquad \bar{\phi}_{okh|l}=v_l\bar{\phi}_{okh}$$

is that the vector field X^i is covariant constant.

The covariant differentiation of (2.7a) yields

$$(2.8) \qquad \bar{\phi}_{jkh|lm}=(v_{l|n}+v_lv_m)\bar{\phi}_{jkh}$$

from (0.20), the equation (2.8) becomes

$$(2.9) \qquad \bar{\phi}_{jkh|lm}=a_{lm}\,\bar{\phi}_{jkh}.$$

Transvecting (2.9) by X^j and using (2.4), we get

$$(2.10) \qquad \bar{\phi}_{okh|lm}=a_{lm}\bar{\phi}_{okh}$$

because X^i is covariant constant for $\bar{\phi}_{jkh}$ to be recurrent.
Hence we state

THEOREM 2.3. *In 2-RGF_n, if the decomposition tensor fields $\bar{\phi}_{jkh}$ and $\bar{\phi}_{okh}$ are first order recurrent then these tensor fields are also second order recurrent along with the condition that the vector field X^i is covariant constant.*

In view of (0.16), (2.1) and (2.4), the Bianchi identity (0.14) takes the form

$$(2.11) \quad X^i[v_l\bar{\phi}_{jkh}+v_k\bar{\phi}_{jhl}+v_h\bar{\phi}_{jlk}]+FX^m[\bar{\phi}_{ohk}\dot{\partial}_m\Gamma^{*i}_{jl}+\bar{\phi}_{ohl}\dot{\partial}_m\Gamma^{*i}_{jk}+\bar{\phi}_{olk}\dot{\partial}_m\Gamma^{*i}_{hj}]$$
$$=2X^i[\bar{\phi}_{jmk}\Delta^m_{[hl]}+\bar{\phi}_{jml}\Delta^m_{[kh]}+\bar{\phi}_{jmh}\Delta^m_{[lk]}]$$

Multiplying (2.11) by v_i and using (2.7a) we have

$$(2.12) \quad \bar{\phi}_{jkh|l}+\bar{\phi}_{jhl|k}+\bar{\phi}_{jlk|k}+FX^mv_i\,[\bar{\phi}_{ohk}\dot{\partial}_m\Gamma^{*i}_{jl}+\bar{\phi}_{ohl}\dot{\partial}_m\Gamma^{*i}_{jk}+\bar{\phi}_{olk}\dot{\partial}_m\Gamma^{*i}_{hj}]$$
$$=2[\bar{\phi}_{jmk}\Delta^n_{[hl]}+\bar{\phi}_{jml}\,\Delta^m_{[kh]}+\bar{\phi}_{jmh}\Delta^m_{[lk]}].$$

Transvecting (2.12) by l^j, we obtain

$$(2.13) \qquad \bar{\phi}_{okh|l}+\bar{\phi}_{ohl|k}+\bar{\phi}_{olk|h}=2[\bar{\phi}_{omk}\Delta^m_{[hl]}+\bar{\phi}_{oml}\,\Delta^m_{[kh]}+\bar{\phi}_{omh}\,\Delta^m_{[lk]}]$$

by virtue of (0.8a) and (2.4).
Accordingly we state

THEOREM 2.4. *In 2-RGF_n the decomposition tensor fields $\bar{\phi}_{jkh}$ and $\bar{\phi}_{okh}$ satisfy the Bianchi identities*

$$[\bar{\phi}_{jkh|l}+\bar{\phi}_{jhl|k}+\bar{\phi}_{jlk|h}]+FX^mv_i[\bar{\phi}_{ohk}\dot{\partial}_m\Gamma^{*i}_{jl}+\bar{\phi}_{ohl}\dot{\partial}_m\Gamma^{*i}_{jk}+\bar{\phi}_{olk}\dot{\partial}_m\Gamma^{*i}_{hj}]$$
$$=2[\bar{\phi}_{jmk}\Delta^m_{[hl]}+\bar{\phi}_{jml}\Delta^m_{[kh]}+\bar{\phi}_{jmh}\Delta^m_{[lk]}]$$

and

$$\bar{\phi}_{okh|l}+\bar{\phi}_{ohl|k}+\bar{\phi}_{olk|h}=2[\bar{\phi}_{omk}\overset{m}{\Delta}_{[hl]}+\bar{\phi}_{oml}\overset{m}{\Delta}_{[kh]}+\bar{\phi}_{omh}\overset{m}{\Delta}_{[lk]}],$$

respectively along with the condition that the vector field X^i is covariant constant.

Differentiating (2.13) covariantly and making use of (2.7b) and (2.10), we obtain

$$(2.14)\quad a_{ln}\bar{\phi}_{okh}+a_{kn}\bar{\phi}_{ohl}+a_{hn}\bar{\phi}_{olk}=v_n v_l\bar{\phi}_{okh}+v_n v_k\bar{\phi}_{ohl}+v_n v_h\bar{\phi}_{olk}+2[\bar{\phi}_{omk}\overset{m}{\Delta}_{[hl]|n}+\bar{\phi}_{oml}\overset{m}{\Delta}_{[kh]|n}+\bar{\phi}_{omh}\overset{m}{\Delta}_{[lk]|n}].$$

From (0.24), the equation (2.14) reduces to

$$(2.15)\quad v_{l|n}\bar{\phi}_{omk}+v_{k|n}\bar{\phi}_{ohl}+v_{h|n}\bar{\phi}_{olk}=2[\bar{\phi}_{omk}\overset{m}{\Delta}_{[hl]|n}+\bar{\phi}_{oml}\overset{m}{\Delta}_{[kh]|n}+\bar{\phi}_{omh}\overset{m}{\Delta}_{[lk]|n}].$$

Hence we state

THEOREM 2.5. *In 2-RGF_n, the decomposition tensor field $\bar{\phi}_{okh}$ satisfies the relation*

$$v_{l|n}\bar{\phi}_{omk}+v_{k|n}\bar{\phi}_{ohl}+v_{h|n}\bar{\phi}_{olk}=2[\bar{\phi}_{omk}\overset{m}{\Delta}_{[hl]|n}+\bar{\phi}_{oml}\overset{m}{\Delta}_{[kh]|n}+\bar{\phi}_{omh}\overset{m}{\Delta}_{[lk]|n}].$$

REMARK 2.1. If the vector field X^i is covariant constant it implies that v_i is also covariant constant.

In view of Remark 2.1, Theorem 2.5 takes the following form: Cor. 2.1. In 2-RGF_n the decomposition tensor field $\bar{\phi}_{okh}$ satisfies the relation

$$\bar{\phi}_{omk}\overset{m}{\Delta}_{[hl]|n}+\bar{\phi}_{oml}\overset{m}{\Delta}_{[kh]|n}+\bar{\phi}_{omh}\overset{m}{\Delta}_{[lk]|n}=0.$$

3. Another decomposition of Cartan's Curvature tensor field K^i_{jkh} in 2-RGF_n

In 2-RGF_n we decompose the Cartan's Curvature tensor field K^i_{jkh} as under

$$(3.1)\qquad K^i_{jkh}=\dot{x}^i\,\overset{*}{\phi}_{jkh},$$

where the decomposition tensor field $\overset{*}{\phi}_{jkh}$ is homogeneous of degree 1 in $\dot{x}^i$. Contracting the indices i and h in (3.1) and using (0.11), we find

$$(3.2)\qquad K_{jk}=\overset{*}{\phi}_{jk},$$

where

$$(3.3)\qquad \overset{*}{\phi}_{jki}\dot{x}^i=\overset{*}{\phi}_{jk}.$$

Transvecting (3.1) by l^j and noting (0.8), we get

(3.4) $$K^i_{okh}=\overset{*}{\phi}_{okh}\dot{x}^i,$$

where

(3.5) $$\overset{*}{\phi}_{okh}=\overset{*}{\phi}_{jkh}l^j$$

Contracting the indices i and h in (3.4), we have

(3.6) $$K_{ok}=\overset{*}{\phi}_{ok},$$

where

(3.7) $$\overset{*}{\phi}_{ok}=\overset{*}{\phi}_{okh}\dot{x}^h.$$

As we know that every RGF_n is also 2-RGF_n hence similar to that of RGF_n [4] we can state following theorems in 2-RGF_n also.

THEOREM 3.1. *In 2-RGF_n the decomposition tensor fields $\overset{*}{\phi}_{jkh}$ and $\overset{*}{\phi}_{okh}$ satisfy the identities*

(3.8) $$\overset{*}{\phi}_{jkh}=-\overset{*}{\phi}_{jhk}$$

and

(3.9) $$\overset{*}{\phi}_{okh}=-\overset{*}{\phi}_{ohk}$$

respectively.

THEOREM 3.2. *In 2-RGF_n, the decomposition tensor field $\overset{*}{\phi}_{ojk}$ is expressed in the form*

(3.10) $$\overset{*}{\phi}_{ojk}=\frac{2}{F}[\overset{*}{\phi}_{[kj]}+\Delta_{[j|k|i],l}\,g^{il}].$$

THEOREM 3.3. *In 2-RGF_n, the decomposition tensor fields $\overset{*}{\phi}_{jkh}$, $\overset{*}{\phi}_{okh}$, $\overset{*}{\phi}_{jk}$ and $\overset{*}{\phi}_{ok}$ behave like recurrent tensor fields as under*

(3.11) $$\overset{*}{\phi}_{jkh|l}=v_l\overset{*}{\phi}_{jkh},$$

(3.12) $$\overset{*}{\phi}_{okh|l}=v_l\overset{*}{\phi}_{okh},$$

(3.13) $$\overset{*}{\phi}_{jk|l}=v_l\overset{*}{\phi}_{jk}$$

and

(3.14) $$\overset{*}{\phi}_{ok|l}=v_l\overset{*}{\phi}_{ok}.$$

THEOREM 3.4. *In 2-RGF_n, the decomposition tensor field $\overset{*}{\phi}_{okh}$ satisfies the* *following Bianchi identity*

$$(3.15)\qquad \overset{*}{\phi}_{okh|l}+\overset{*}{\phi}_{chl|k}+\overset{*}{\phi}_{olk|h}=2[\overset{*}{\phi}_{cml}\Delta^{m}_{[kh]}+\overset{*}{\phi}_{cmk}\Delta^{m}_{[hl]}+\overset{*}{\phi}_{cmh}\Delta^{m}_{[lk]}].$$

In view of Theorem 3.3 and the equation (0.24) we state

THEOREM 3.5. *In 2-RGF_n the decomposition tensor fields $\overset{*}{\phi}_{jkh}$, $\overset{*}{\phi}_{okh}$, $\overset{*}{\phi}_{jk}$ and $\overset{*}{\phi}_{ok}$ are also second order recurrent i.e.*

$$(3.16)\ \text{(a)}\qquad \overset{*}{\phi}_{jkh|lm}=a_{lm}\overset{*}{\phi}_{jkh},$$

$$\text{(b)}\qquad \overset{*}{\phi}_{okh|lm}=a_{lm}\overset{*}{\phi}_{okh},$$

$$\text{(c)}\qquad \overset{*}{\phi}_{jk|lm}=a_{lm}\overset{*}{\phi}_{jk},$$

$$\text{(d)}\qquad \overset{*}{\phi}_{ok|lm}=a_{lm}\overset{*}{\phi}_{ok}.$$

The covariant differentiation of (3.13) yields

$$(3.17)\qquad a_{[lm]}\overset{*}{\phi}_{jk}=\overset{*}{\phi}_{jk|[lm]}$$

in view of (0.24)

Using commutation formula (0.6), the equation (3.1), (3.4) and (3.13) in (3.17), we find

$$(3.18)\qquad 2a_{[lm]}\overset{*}{\phi}_{jk}=\overset{*}{\phi}_{jk}\overset{*}{\phi}_{olm}F-\overset{*}{\phi}_{hk}\dot{x}^{h}\overset{*}{\phi}_{jml}-\overset{*}{\phi}_{jh}\dot{x}^{h}\overset{*}{\phi}_{kml}-2\overset{*}{\phi}_{jk}v_{h}\Delta^{h}_{[lm]}.$$

Transvecting (3.18) by l^{j} and noting (3.5), we have

$$(3.19)\qquad 2a_{[lm]}\overset{*}{\phi}_{ok}=2\overset{*}{\phi}_{ok}\phi_{olm}F-\overset{*}{\phi}_{oh}\dot{x}^{h}\overset{*}{\phi}_{kml}-2\overset{*}{\phi}_{ok}v_{h}\Delta^{h}_{[lm]},$$

where $\overset{*}{\phi}_{ok}=\overset{*}{\phi}_{jk}$ and $l^{j}=\dot{x}^{j}|F$.

If we assume $\overset{*}{\phi}_{oh}\dot{x}^{h}=0$, the equation (3.19) becomes

$$(3.20)\qquad \overset{*}{\phi}_{olm}=\frac{1}{F}[a_{[lm]}+v_{h}\Delta^{h}_{[lm]}]$$

because $\overset{*}{\phi}_{ok}\neq 0$.

Conversely if (3.20) is true, the equation (3.19) reduces to

$$(3.21)\qquad \overset{*}{\phi}_{oh}\dot{x}^{h}\overset{*}{\phi}_{kml}=0.$$

Since $\overset{*}{\phi}_{kml}\neq 0$, therefore (3.21) takes the form

$$(3.22)\qquad \overset{*}{\phi}_{oh}\dot{x}^{h}=0.$$

Accordingly we have

THEOREM 3.6. *In 2-RGF_n, the necessary and sufficient condition for the decomposition tensor field $\overset{*}{\phi}_{olm}$ to be expressed as*

$$\overset{*}{\phi}_{olm}=F^{-1}[a_{[lm]}+v_h\Delta^h_{[lm]}]$$

is that

$$\overset{*}{\phi}_{oh}\dot{x}^h=0.$$

Taking covariant differentiation of (3.15) and making use of (3.12), (3.15) and (3.16b), we obtain

$$(3.23)\qquad a_{lm}\overset{*}{\phi}_{okh}+a_{kn}\overset{*}{\phi}_{ohl}+a_{hn}\overset{*}{\phi}_{olk}=v_lv_n\overset{*}{\phi}_{okh}+v_kv_n\overset{*}{\phi}_{ohl}+v_hv_n\overset{*}{\phi}_{olk}+2[\overset{*}{\phi}_{oml}\Delta^m_{[kh]|n}+\overset{*}{\phi}_{omk}\Delta^m_{[hl]|n}+\overset{*}{\phi}_{omh}\Delta^m_{[lk]|n}].$$

Simplifying with help of (0.24), the equation (3.23) gives

$$(3.24)\qquad v_{l|n}\overset{*}{\phi}_{okh}+v_{k|n}\overset{*}{\phi}_{ohl}+v_{h|n}\overset{*}{\phi}_{olk}=2[\overset{*}{\phi}_{oml}\Delta^m_{[kh]|n}+\overset{*}{\phi}_{omk}\Delta^m_{[hl]|n}+\overset{*}{\phi}_{omh}\Delta^m_{[lk]|n}].$$

We state,

THEOREM 3.7. *In 2-RGF_n, the relation*

$$v_{l|n}\overset{*}{\phi}_{okh}+v_{k|n}\overset{*}{\phi}_{ohl}+v_{h|n}\overset{*}{\phi}_{olk}=2[\overset{*}{\phi}_{oml}\Delta^m_{[kh]|n}+\overset{*}{\phi}_{omk}\Delta^m_{[hl]|n}+\overset{*}{\phi}_{omh}\Delta^m_{[lk]|n}]$$

holds good.

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