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BIRECURRENT GENERALISED FINSLER SPACES

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Introduction

Author and Sinha [2] have defined recurrent generalised Finsler space of first order and studied the properties of recurrent curvature tensor field and recurrence vector field. The object of present paper is to define birecurrent generalised Finsler spaces. Author has also discussed the properties of recurrence tensor and recurrent curvature tensor fields in these spaces.

We consider an n-dimensional Finsler space F_n endowed with a local coordinate system x^i. The metric tensor $g_{ij}(x,\dot{x})$ of F_n is considered here as non-symmetric in general. The spaces endowed with this metric tensor are known as generalised Finsler spaces and we denote them by GF_n.

The connection parameters for the locally Minkowskian and locally Euclidean GF_n are denoted by P^{*i}_{jk} and Γ^{*i}_{jk} respectively. Let T^i be a vector field of GF_n, then the two processes of differentiation are defined as follows:

$$T^i_{,j}=\partial_j T^i+\partial_j \dot{x}^h \partial_h T^i+P^{*i}_{kj}T^k \tag{1.1}$$

and

$$T^i|_j=\partial_j T^i-\Gamma^h_{kj}\dot{\partial}_h T^i \dot{x}^k+\Gamma^{*i}_{kj}T^k, \tag{1.2}$$

where

$$\Gamma^i_{jk}=\Gamma^{*i}_{jk}+C^i_{jh}\Gamma^{*h}_{rk}\dot{x}^r \tag{1.3}$$

and

$$C_{ijk}=\frac{1}{4}\dot{\partial}^3_{ijk}F^2(x,\dot{x}). \tag{1.4}$$

With the help of above covariant differentiations two curvature tensor fields $\bar{K}^i_{jhk}$ and K^i_{jkh} are defined. The commutation formular involving these curvature tensor fields are given as under [1]:

1) The numbers in brackets refer to the references.

2) $2T^i_{,[jk]}=T^i_{,jk}-T^i_{,kj}$.

(1.5) $$2T^i_{,[jk]}=T^h\bar{K}^i_{hkj}-2T^i_{,h}\Delta^h_{[jk]}$$ 2)

and

(1.6) $$2T^i|_{[jk]}=\dot{\partial}_h T^i K^h_{ojk}F+T^h K^i_{hkj}-2T^i|_h\Delta^h_{[jk]},$$

where

(1.7) $$\Gamma^{*i}_{[jk]}=P^{*i}_{[jk]}=\Delta^i_{[jk]}$$

and

(1.8) $$K^i_{okh}=K^i_{jkh}l^j.$$

In GF_n, the curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} satisfy the following identities

(1.9) $$\bar{K}^i_{jkh}=-\bar{K}^i_{jhk},\ K^i_{jkh}=-K^i_{jhk},$$

(1.10) $$\bar{K}^i_{jkh}+\bar{K}^i_{khj}+\bar{K}^i_{hjk}=2\Delta_{[j|k|h];l}\,g^{il},$$

where (;) denotes covariant derivative based upon the connection parameter given by

(1.12) $$\bar{K}^i_{jkh,l}+\bar{K}^i_{jhl,k}+\bar{K}^i_{jlk,h}+2[\bar{K}^i_{jmk}P^{*m}_{[lh]}+\bar{K}^i_{jmh}P^{*m}_{[kl]}+\bar{K}^i_{jml}P^{*m}_{[hk]}=0$$

and

(1.13) $$K^i_{jkh}|_l+K^i_{jhl}|_k+K^i_{jlk}|_h+F[K^m_{ohk}\dot{\partial}_m\Gamma^{*i}_{jl}+K^m_{olh}\dot{\partial}_m\Gamma^{*i}_{jk}+K^m_{okl}\dot{\partial}_m\Gamma^{*i}_{hj}]$$
$$=2[K^i_{jml}\Delta^m_{[kh]}+K^i_{jmk}\Delta^m_{[hl]}+K^i_{jmh}\Delta^m_{[lk]}].$$

Here $g_{(ij)}$ represents symmetric parts and g^{ij} is conjugate tensor of $g_{(ij)}$.

Author and Sinha [2] have defined recurrent curvature tensor field in GF_n as under:

The GF_n, in which there exists a non-zero vector v_l such that the curvature tensor fields $\bar{K}^i_{jkh}$ and K^i_{jkh} satisfy the relations

(1.14) $$\bar{K}^i_{jkh,l}=v_l\bar{K}^i_{jkh}$$

and

(1.15) $$K^i_{jkh}|_l=v_lK^i_{jkh}$$

respectively, are said to be *recurrent* GF_n (or in brief RGF_n) and the curvature tensor fields of these spaces are called *recurrent curvalure tensor ieldsf*. Here v_l is known as recurrence vector field.

Contracting the indices i and h in (1.15), we find

(1.16) $$K_{jk}|_l = v_l K_{jk}$$

Transvecting (1.15) by l^j and using (1.8), we write

(1.17) $$K^i_{okh}|_l = v_l K^i_{okh}$$

2. Birecurrent relative curvature tensor field

DEFINITIONS. 1. An n-dimensional generalised Finsler space GF_n, in which relative curvature tensor field $\bar{K}^i_{jkh}$ satisfies the relation

(2.1) $$\bar{K}^i_{jkh,\ lm} = a_{lm}\bar{K}^i_{jkh},\quad \bar{K}^i_{jkh} \neq 0,$$

where a_{lm} is a *non-zero recurrence tensor field,* is defined as birecurrent generalised Finsler space. It is denoted by $2\text{-}RG\tilde{F}_n$.

2. The relative curvature tensor field $\bar{K}^i_{jkh}$ of $2\text{-}RG\tilde{F}_n$ which satisfies (2.1), is called *birecurrent relative curvature tensor field.*

Some theorems on recurrence tensor field a_{lm} and birecurrent relative curvature tensor field $\bar{K}^i_{jkh}$ are established in this section.

The covariant differentiation of (1.14) with respect to x^m yields

(2.2) $$\bar{K}^i_{jkh,\ lm} = (v_{l,\ m} + v_l v_m)\bar{K}^i_{jkh}$$

(2.3) $$a_{lm} = (v_{l,\ m} + v_l v_m).$$

Thus we state

THEOREM 2.1. *Every generalised Finsler space of first order for which recurrence vector field v_l satisfies*

$$v_{l,\ m} + v_l v_m \neq 0$$

is also a birecurrent generalised Finsler space but the converse is not true in general.

Commuting the indices l and m in (2.1) and subtracting the result thus obtained from it, we get

(2.4) $$\bar{K}^i_{jkh,\ lm} - \bar{K}^i_{jkh,\ ml} = (a_{lm} - a_{ml})\bar{K}^i_{jkh}$$

By virtue of commutation formula (1.5), the equation (2,4) takes the form

(2.5) $$\bar{K}^i_{rml}\bar{K}^r_{jkh} - \bar{K}^i_{rkh}\bar{K}^r_{jml} - \bar{K}^i_{jrh}\bar{K}^r_{kml} - \bar{K}^i_{jkr}\bar{K}^r_{hml} - 2\bar{K}^i_{jkh,\ r}\Delta^r_{[lm]} = 2a_{lm}\bar{K}^i_{jkh}.$$

Thus we have

THEOREM 2.2. *In $2\text{-}RG\tilde{F}_n$, the recurrence tensor field is not symmetric in* *general.*

Differentiating (2.5) covariently with respect to x^n and simplifying with the help of (1.14), (2.1) and (2.5), we have

(2.6) $v_n a_{[lm]}\bar{K}^i_{jkh}+2v_n v_r \bar{K}^i_{jkh}\Delta^r_{[lm]} = a_{[lm],n}\bar{K}^i_{jkh}+a_{rn}\Delta^r_{[lm]}\bar{K}^i_{jkh}+v_r\bar{K}^i_{jkh}\Delta^r_{[lm],n}$,

which yields

(2.7) $v_n a_{[lm]}+v_n v_r\Delta^r_{[lm]}=v_{r,n}\Delta^r_{[lm]}+a_{[lm],n}+v_r\Delta^r_{[lm],n}$ since $\bar{K}^i_{jkh}\neq 0$.

Accordingly we have

THEOREM 2.3. *In 2-$RG\tilde{F}_n$, the recurrence tensor field satisfies the relation*

$$v_n a_{[lm]}+v_n v_r\Delta^r_{[lm]}=v_{r,n}\Delta^r_{[lm]}+a_{[lm],n}+v_r\Delta^r_{[lm],n}$$

The covariant differentiation of (2.7) with respect to x^s gives

(2.8) $v_{n,s}a_{[lm]}+v_{n,s}v_r\Delta^r_{[lm]}+v_n v_s a_{[lm]}+v_n v_s v_r\Delta^r_{[lm]}=a_{[lm],ns}+v_{r,ns}\Delta^r_{[lm]}$
$+v_{r,n}\Delta^r_{[lm],s}+v_{r,s}\Delta^r_{[lm],n}+v_r\Delta^r_{[lm],ns}$

Noting (2.3), the equation (2.8) becomes

(2.9) $a_{ns}a_{[lm]}+a_{ns}v_r\Delta^r_{[lm]}=a_{[lm],ns}+v_{r,ns}\Delta^r_{[lm]}+v_{r,n}\Delta^r_{[lm],s}+v_{r,s}\Delta^r_{[lm],n}$
$+v_r\Delta^r_{[lm],ns}$

Commuting the indices n and s in (2.9) and subtracting the obtained result from it, we get

(2.10) $a_{[ns]}(a_{[lm]}+v_r\Delta^r_{[lm]})=a_{[lm],[ns]}+v_{r,[ns]}\Delta^r_{[lm]}+v_r\Delta^r_{[lm],[ns]}$

Hence we have

THEOREM 2.4. *In 2-$RG\tilde{F}_n$, the relation*

$$a_{[ns]}(a_{[lm]}+v_r\Delta^r_{[lm]})=a_{[lm],[ns]}+v_{r,[ns]}\Delta^r_{[lm]}+v_r\Delta^r_{[lm],[ns]}$$

is true.

By virtue of commutation formula (1.5) and (1.9), the equation (2.10) yields

(2.11) $2a_{[ns]}(a_{[lm]}+v_r\Delta^r_{[lm]})=a_{[rn]}\bar{K}^r_{lns}+a_{[lr]}\bar{K}^r_{mns}-2\Delta^r_{[ns]}a_{[lm],r}+v_p\bar{K}^p_{rns}\Delta^r_{[lm]}$
$-2v_{r,p}\Delta^p_{[ns]}\Delta^r_{[lm]}+2\Delta^r_{[lm],[ns]}v_r$

Taking cyclic permutation of the indices l, n and s in (2.11) and noting (1.10), we obtain

$$(2.12)\quad a_{[\overline{ns}}a_{\overline{l]m}}+a_{[\overline{ns}}\Delta^{r}_{\overline{l]m}}v_r=a_{\overline{rm}}\Delta_{[l|n|s];q}g^{rq}+\frac{1}{2}\bar{K}^{r}_{m[ns}a_{\overline{l]r}}-\Delta^{r}_{[\overline{ns}}a_{\overline{l]m},r}$$
$$+\frac{1}{2}v_p\bar{K}^{p}_{r[ns}\Delta^{r}_{\overline{l]m}}-v_{r,p}\Delta^{p}_{[\overline{ns}}\Delta^{r}_{\overline{l]m}}+v_r\Delta^{r}_{[\overline{l|m|},\ \overline{ns}]},$$

where bar over indices represents skew-symmetric part and the index between box is unaltered under cyclic permutation of indices. Accordingly we state:

THEOREM 2.5. *In* $2-RG\tilde{F}_n$

$$a_{[\overline{ns}}a_{\overline{l]m}}+a_{[\overline{ns}}\Delta^{r}_{\overline{l]m}}v_r=a_{\overline{rm}}\Delta_{[l|n|s];q}g^{rq}+\frac{1}{2}\bar{K}^{r}_{m[ns}a_{\overline{l]r}}-\Delta^{r}_{[\overline{ns}}a_{\overline{l]m},r}$$
$$+\frac{1}{2}v_p\bar{K}^{p}_{r[ns}\Delta^{r}_{\overline{l]m}}-v_{r,p}\Delta^{p}_{[\overline{ns}}\Delta^{r}_{\overline{l]m}}+v_r\Delta^{r}_{[\overline{l|m|},\ \overline{ns}]}$$

The covariant differentiation of (1.12) with respect of x^n gives

$$(2.13)\quad a_{ln}\bar{K}^{i}_{jkh}+a_{kn}\bar{K}^{i}_{jhl}+a_{hn}\bar{K}^{i}_{jlk}+2v_n[\bar{K}^{i}_{jmk}P^{*m}_{[lh]}+\bar{K}^{i}_{jmh}P^{*m}_{[lk]}+\bar{K}^{i}_{jml}P^{*m}_{[hk]}]$$
$$+2[\bar{K}^{i}_{jmk}P^{*m}_{[lh],n}+\bar{K}^{i}_{jmh}P^{*m}_{[lk],n}+\bar{K}^{i}_{jml}P^{*m}_{[hk],n}]=0$$

by means of (1.14) and (2.1).

From (1.12) and (1.14), the equation (2.13) takes the form

$$(2.14)\quad (a_{ln}-v_nv_l)\bar{K}^{i}_{jkh}+(a_{kn}-v_nv_k)\bar{K}^{i}_{jhl}+(a_{hn}-v_hv_n)\bar{K}^{i}_{jlk}+2[\bar{K}^{i}_{jmk}p^{*m}_{[lh],n}$$
$$+\bar{K}^{i}_{jmh}P^{*m}_{[lk],n}+\bar{K}^{i}_{jml}P^{*m}_{[hk],n]}]=0$$

In view of (2.3), above equation reduces to

$$(2.15)\qquad \bar{K}^{i}_{j[kh}v_{l],n}-2\bar{K}^{i}_{jm[k}P^{*m}_{\overline{hl}],n}=0$$

Thus we write

THEOREM 2.6. *In* $2-RG\tilde{F}_n$, *the Bianchi identity takes the form*

$$\bar{K}^{i}_{j[kh}v_{l],n}-2\bar{K}^{i}_{jm[k}P^{*m}_{\overline{hl}],n}=0$$

3. Birecurrent Castan's Curvature tensor field

DEFINITIONS. 1. An n-dimensional generalised Finsler space GF_n, in which Cartan's curvature tensor field K^{i}_{jkh} satisfies the relation

$$(3.1)\qquad K^{i}_{jkh}|_{lm}=a_{lm}K^{i}_{jkh},\ \ K^{i}_{jkh}\neq 0,$$

where a_{lm} is a non-zero recurrence tensor field, is called *birecurrent generalised* Finsler space. It is denoted by 2-RGF_n in brief.

2. The curvature tensor field K^{i}_{jkh} which satisfies (3.1) is defined as birecur

rent curvature tensor field.

Transvecting (3.1) by l^j, we find

(3.2) $$K^i_{okh}|_{lm}=a_{lm}K^i_{okh}$$

by means of (1.8).

Contracting the indices i and h in (3.1), we have

(3.3) $$K_{jk}|_{lm}=a_{lm}K_{jk}$$

along with $K_{jk}\neq 0$.

Taking covariant differentiation of (1.15) with respect to x^m, we get

(3.3) $$K^i_{jkh}|_{lm}=(v_{l|m}+v_l v_m)K^i_{jkh}$$

by virtue of (1.15).

From (3.1) and (3.3), we write

(3.4) $$a_{lm}=v_l|_m+v_l v_m$$

Hence we have

THEOREM 3.1. *Every recurrent generalised Finsler space of first order in the sense of Cartan for which recurrence vector field* v_l *satisfies* $v_l|_m+v_l v_m\neq 0$ *is also a* 2-RGF_n *but the converse is not true in general.*

Similar to that of Theorem 2.2, we state

THEOREM 3.2. *In* 2-RGF_n, *the recurrence tensor field* a_{lm} *is not symmetric in general.*

Commuting the indices l and m in (3.3) and subtracting the result from it, we obtain

(3.5) $$K_{jk}|_{lm}-K_{jk}|_{ml}=(a_{lm}-a_{ml})K_{jk}.$$

Using Commutation formula (1.6) and equation (1.16) in (3.5), we get

(3.6) $$\dot{\partial}_h K_{jk}K^h_{olm}F-K_{pk}K^p_{jml}-K_{jp}K^p_{kml}-2K_{jk}|_h\Delta^h_{[lm]}=(a_{lm}-a_{ml})K_{jk}$$

Differentiating (3.6) covariantly with respect to x^n and noting (1.15), (1.16), (1.7) and (3.6) we have

(3.7) $$(\dot{\partial}_h K_{jk})|_n K^h_{olm}F-v_n(K_{pk}K^p_{jml}+K_{jp}K^p_{kml})=(2a_{[lm]}|_n+2v_h|_n\Delta^h_{[lm]}+2v_h\Delta^h_{[lm]}|_n)K_{jk}$$

Thus we state

THEOREM 3.3. *In* 2-RGF_n, *the relation*

$$(\dot{\partial}_h K_{jk})|_n K^h_{olm} F - v_n(K_{pk}K^p_{jml} + K_{jp}K^p_{kml}) = (2a_{[lm]}|_n + v_h|_n \Delta^h_{[lm]} + 2v_h \Delta^h_{[lm]}|_n) K_{jk}$$

holds good.

The covariant differentiation of (1.13) with respect to x^n yields

$$(3.8)\quad a_{ln} K^i_{jkh} + a_{kn} K^i_{jhl} + a_{hn} K^i_{jlk} + F v_n [K^m_{ohk} \dot{\partial}_m \Gamma^{*i}_{jl} + K^m_{olh} \dot{\partial}_m \Gamma^{*i}_{jk} + K^m_{okl} \dot{\partial}_m \Gamma^{*i}_{hj}]$$
$$+ F [K^m_{ohk} (\dot{\partial}_m \Gamma^{*i}_{jl})|_n + K^m_{olh} (\dot{\partial}_m \Gamma^{*i}_{jk})|_n + K^m_{okl} (\dot{\partial}_m \Gamma^{*i}_{hj})|_n] = 2v_n [K^i_{jml} \Delta^m_{[kh]}$$
$$+ K^i_{jmk} \Delta^m_{[hl]} + K^i_{jmh} \Delta^m_{[lk]}] + 2[K^i_{jml} \Delta^m_{[kh]}|_n + K^i_{jmk} \Delta^m_{[hl]}|_n + K^i_{jmh} \Delta^m_{[lk]}|_n]$$

in view of (1.15) (1.17) and (3.1).

Simplifying (3.8) with the help of (1.13) and (3.4) it gives

$$(3.9)\quad v_l|_n K^i_{jkh} + v_k|_n K^i_{jhl} + v_h|_n K^i_{jlk} + F [(\dot{\partial}_m \Gamma^{*i}_{jl})|_n K^m_{ohk} + K^m_{olk} (\dot{\partial}_m \Gamma^{*i}_{jk})|_n$$
$$+ K^m_{okl} (\dot{\partial}_m \Gamma^{*i}_{hj})|_n] = 2[K^i_{jml} \Delta^m_{[kh]}|_n + K^i_{jmk} \Delta^m_{[hl]}|_n + K^i_{jmh} \Delta^m_{[lk]}|_n]$$

which takes the form

$$(3.10)\quad K^i_{j[kh} v_{l]}|_n - F K^m_{o[kh} \dot{\partial}_{\boxed{m}} \Gamma^{*i}_{\boxed{j} l]}|_n = 2K^i_{jm[l} \Delta^m_{\overline{kh}]}|_n$$

by virtue of (1.9). The indices between boxes are unaltered under cyclic change of indices and bar represents represents skew-symmetric part.

Accordingly we have

THEOREM 3.4. *In 2-RGF_n, the Bianchi identity for the curvature tensor field* K^i_{jkh} takes the form

$$K^i_{j[kh} v_{l]}|_n - F K^m_{o[kh} \dot{\partial}_{\boxed{m}} \Gamma^{*i}_{\boxed{j} l]}|_n = 2K^i_{jm[l} \Delta^m_{\overline{kh}]}|_n$$

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