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CONDITIONS FOR RINGS OF TYPE-A TO BE BOOLEAN

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In solving a problem proposed by D. Jacobson [1], E. Wong, among other solvers, proved in [4] that if R is a commutative regular ring and 1 is the only unit in R, then R is a Boolean ring. And this result can be extended to a class of associative (not necessarily commutative) rings. Later, H. Myung proved in [2] that an alternative ring R with identity 1 is a Boolean ring if and only if R is von Neumann regular and 1 is the only unit. Recently R.A. Melter [2] proposed the following problem: in which rings is the following proposition valid: $x=y$ if and only if $(1-x+xy)(1-y+yx)=1$? In this note, we shall prove that if R is an alternative ring and 1 is the only unit, then the condition $x=y$ iff $(1-x+xy)(1-y+yx)=1$ holds in R if and only if R is von Neumann regular, i.e. a Boolean ring.

Let R be a ring, not necessarily associative or commutative. R is said to be of *type-A* if, for an idempotent e and an element a in R, the subalgebra of R generated by e and a is associative. It is clear that the class of rings of type-A is a generalization of the class of alternative rings in the sense that the subalgebra of an alternative ring generated by any two elements is associative [Artin's theorem]. Thus, all associative rings are of type-A.

The following definition is an appropriate extension of regular rings for a wider class of rings.

DEFINITION. A ring R is said to be *strongly regular* if for each $a\in R$ there exists an element b in R such that $(ab)a=a$ and the subalgebra of R generated by a and b is associative.

It is clear that for alternative rings or associative rings, the concept of strong regularity is identical with that of the usual regularity. The following lemma is a slight modification of a well known fact.

LEMMA. *Let R be a unitary ring of type-A without nonzero nilpotents. If the elements a and b of R satisfy $(ab)a=a$ and the subalgebra of R generated by a and b is associative, then there is a unit element s in R such that $(as)a=a$.*

THEOREM. *Let R be a unitary ring of type-A. The following statements are* *equivalent*:

(1) *R is Boolean.*

(2) *For x, $y \in R$, $x=y$ iff $(1-x+xy)(1-y+yx)=1$ and $a^2=1$ holds for only $a=1$.*

(3) *R is strongly regular and 1 is the only unit.*

PROOF. (1)$\Longrightarrow$(2). The second part is trivial. Suppose $x=y$. Then $(1-x+xy)(1-y+yx)=(1-x+x^2)^2=1$. Conversely, let $(1-x+xy)(1-y+yx)=1$. We show that every unit is equal to 1. Let u be a unit in R. Since R is of type-A, the subalgebra generated by u and u^{-1} is associative. Thus, $1=u^{-1}u=u^{-1}u^2=u^{-1}(u\ u)=(u^{-1}\ u)u=u$. This implies $1-x+xy=1$ and $1-y+yx=1$. That is $x=xy=yx=y$.

(2)$\Longrightarrow$(3). Let $x \in R$. Then $(1-x+x^2)^2=1$, and hence $1-x+x^2=1$. Thus $x^2=x$, i.e. x is an idempotent. Now, we show that if $(xy)x=x$ for some $y \in R$, then the subalgebra generated by x and y is associative. This is in fact clear, because R is of type-A and both x and y are idempotent. That is, R is strongly regular. Let u be a unit. Then $u^2=u$. Again, $u=(u^{-1}u)u=u^{-1}u^2=u^{-1}u=1$.

(3)$\Longrightarrow$(1). Let $a^2=0$ in R. Then $1=1-a^2=(1-a)(1+a)$. This implies that $1-a=1$, and hence $a=0$. That is, the ring R has no non-zero nilpotents. By the Lemma, for each $x \in R$, there exists a unit s in R such that $(xs)x=x$. But $s=1$. That is, $x^2=x$. This completes the proof.

COROLLARY. *Let R be a unitary ring of type-A. Then R is Boolean iff it is alternative regular and 1 is the only unit in R.*

REMARK. There is a class of rings that are far different from being type-A yet satisfying the conditions of the theorem. We consider the algebra R over Z_2 generated by the set $\{1, s_1, s_2, s_3\}$ with the following operation, $s_i^2=s_i (i=1, 2, 3)$, $s_1=s_1s_i=s_is_1 (i=2, 3)$ and $s_2s_3=s_3s_2=0$. This ring has been considered in [2], and it is claimed to be not Boolean. However, one notes that the requirement for a ring to be Boolean is only the idempotency $a^2=a$. Therefore, a Boolean ring is not assumed to be associative. In fact, the ring defined above is a non-associative Boolean ring in this context. Now we observe that the ring satisfies all the conditions in the theorem, but unfortunately it is not of type-A. Suppose it was of type-A. Then by the corollary it would be alternative. But the ring is not alternative. For example, let $a=s_1+s_2$ and $b=s_2+s_3$. Then $ab=$

s_2, and thus $a(ab)=s_1+s_2$. On the other hand, $a^2b=ab=s_2$. That is, $a^2b\neq a(ab)$.

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