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LINE COMPLEXES IMMERSED IN A PROJECTIVE SPACE WITH RULED ABSOLUTE

By M. A. Soliman and N. H. Abdel

1. Introduction

We consider an three-dimensional projective space P_3 referred to a moving frame $\{A_i\}$ of four linearly independent analytic points A_1, A_2, A_3, A_4. An infinitesimal displacement of such a frame is determined by the equations,

$$dA_i=\omega_i^j A_j \qquad (i,\ j,\ X=1,\ \cdots,\ 4), \tag{1.1}$$

where the one-forms ω_i^j (Pfaff's differential forms) are invariant one-forms of the projective group $PG(3, R)$ whose structural equations have the form

$$D\ \omega_i^j=\omega_i^k\wedge\omega_k^j \tag{1.2}$$

We consider the geometry belonging to a subgroup H_1^3 of the group $PG(3, R)$, the transformations in the subgroup H_1^3 do not move a ruled surface σ. In [1], it was shown that in a partially canonical moving frame $\{A_i\}$, the ruled surface (absolute) σ is determined by the equation

$$x^1x^2-x^3x^4=0 \tag{1.3}$$

where the points A_1, A_2, A_3, A_4 are located on σ and $(A_1, A_2, A_3, A_4)=1$. The conditions of the stationary subgroup H_1^3 of the projective group $PG(3, R)$ are

$$\left.\begin{array}{l}\omega_1^2=\omega_2^1=0,\ \ \omega_4^3=\omega_3^4=0,\ \ \omega_3^2-\omega_1^4=0,\ \ \omega_2^3-\omega_4^1=0\\ \omega_1^3-\omega_4^2=0,\ \ \omega_3^1-\omega_2^4=0,\ \ \omega_1^1+\omega_2^2=0,\ \ \omega_3^3+\omega_4^4=0\end{array}\right] \tag{1.4}$$

From (1.1) and (1.4) it follows that, there exist two families of generators $\Phi_1=\{A_2A_3, A_1A_4\}$ and $\Phi_2=\{A_1A_3, A_2A_4\}$ of the absolute σ.

The set of all lines in the space P_3 is called the *Grassman manifold* $Gr(1,3)$. It is well known dim $Gr(1,3)=4$. A smooth r-dimensional submanifold will be denoted by $Gr(1, 3, r)$ $(1\leq r\leq 4)$.

DEFINITION 1.1. A three-dimensional (two-dimensional) submanifold of the Grassman manifold $Gr(1,3)$ is called a *line complex* $Gr(1,3,3)$ (*a line congruence*

$Gr(1,3,2)\}$ immersed in P_3.

2. Line complexes embedded in P_3 with ruled absolute

Let the points A_1, A_2 be located on a moving straight line l of a line complex $Gr(1,3,3)$. Then the invariance condition of l under an infinitesimal transformations of the subgroup H_1^3 is of the form

$$\omega_p^\alpha=0 \ (p=1,2;\ \alpha=3,4).$$

Hence one-forms ω_p^α are the main forms of the Grassman manifold $Gr(1, 3)$. Thus, in such a frame the differential equation of a line complex $Gr(1, 3, 3)$ have the form

$$\omega_1^3=a \ \ \omega_2^3+b \ \ \omega_1^4+\chi\omega_2^4 \tag{2.1}$$

in the first-order contact element of the generating element l of this complex.

Expanding the exterior quadratic equations corresponding to (2.1) by Cartan's lemma, we get that the equations for the infinitesimal variations of the quantities a, b, χ when the first-order parameters are fixed, are as follows

$$\{\delta a \ \delta b \ \delta\chi\}^T=M\{\pi_1^1 \pi_3^3\}^T \tag{2.2}$$

where $M=\begin{bmatrix} 2a & 0 \\ 0 & -2b \\ 2\chi & -2\chi \end{bmatrix}$, T denotes matrix transposition, δ is the symbol for differentiation with respect to the second order parameters and $\pi_i^j=\omega_i^j(\delta)$.

The system of quantities $\theta=\{a, b, \chi\}$ forms the first fundamental differential geometric object of the manifold $G_r(1,3,3)$ [5].

Therein, we give a geometric interpretation of the geometric object θ. Since the lin $l=A_1A_2$ describes the line complex (2.1), we have the normal correspondence

$$K: N(t)=A_1+tA_2 \longleftrightarrow \Sigma(N(t)): x^3-\lambda x^4=0,$$

where

$$\lambda=(t+a)/(\chi-bt) \tag{2.3}$$

We thus obtain a projective mapping of the points of the line l onto the sheaf of planes $x^3-\lambda x^4=0$. This mapping associates the invariant point $N_1=bA_1+\chi A_2$ with the plane $x^4=0$. The point N_1 together with $N_2=bA_1-\chi A_2$, harmonically separates the pair of points A_1 and A_2. If we put $t=0$ or $t=\infty$, then the projective mapping (2.3) will determine two planes

$$\chi x^3 - ax^4 = 0 \tag{2.4}$$

$$bx^3 + x^4 = 0 \tag{2.5}$$

from the sheaf of planes $x^3 - \lambda x^4 = 0$. These two planes correspond to the points A_1 and A_2. From (2.2), we get

$$\delta(1/b) = 2(1/b)\, \pi_3^3, \quad \delta(a/\chi) = 2(a/\chi)\, \pi_3^3 \tag{2.6}$$

since equations (2.6) are of form analogous to the equation $\delta\lambda = 2\lambda\pi_3^3$, which follows from the stationarity of any plane of the sheaf $x^3 - \lambda x^4 = 0$, we conclude that the invariant planes (2.4) and (2.5) belong to the sheaf $x^3 - \lambda x^4 = 0$.

DEFINITION 2.1. [6] A Grassman manifold $Gr(1,3)$ with a field of correspondence K is called a *nonholonomic complex* NGr (1, 3, 3) which determined by (2.1).

Generally, the matrix M has rank equal to two. For the general class of line complexes $NGr(1,3,3)$ and from (2.2) we have

$$\delta\chi = \chi(b\delta a + a\delta b)/(ab),$$

i.e., we can take $\chi = ab$. This class of line complexes is defined by the system of linear differential equation

$$\omega_1^3 - a\ \ \omega_2^3 = b\left(\omega_1^4 + a\omega_2^4\right) \tag{3.7}$$

and the associated exterior quadratic equations

$$da - 2a\omega_1^1 = \mu_1\left(\omega_2^3 + b\,\omega_2^4\right) + \mu_2\left(\omega_1^4 + a\,\omega_2^4\right)$$

$$db + 2b\omega_3^3 = \mu_2\left(\omega_2^3 + b\,\omega_2^4\right) + \mu_3\left(\omega_1^4 + a\omega_2^4\right)$$

where μ_1, μ_2, μ_3 are the invariants in the *2nd*-order contact element. This is an involutive system, the nonuniqueness of its solution being characterized by one function of two arguments. Hence we have the following lemma.

LEMMA 2.1. *The range of existence of a line complex* $NGr(1,3,3)$ *embedded in* P_3 *with ruled absolute comprises one arbitrary function of two arguments.*

We state the results of our study of the geometry of two special classes of line complexes of the above type in the following sections [7].

3. Class of the holonomic line complexes $HGr_3(1,3,3)$

In this section, we consider the special class of line complexes $NGr(1,3,3)$ in which the rank of the matrix M equal to zero, i.e., $a=b=\chi=0$. We denote the resulting manifold by $NGr_3(1,3,3)$ {the lower index indicates the number of coefficients here which are equal to zero in equation (2.1)}. Since the geometric object θ is empty and the normal correspondence is degenerate, the line complex $NGr_3(1,3,3)$ become a holonomic line complex $HGr_3(1,3,3)$.

The line complex $HGr_3(1,3,3)$ is defined by the Pfaffian equation

$$\omega_1^3=0 \tag{3.1}$$

This is an involutive equation, the uniqueness of its solution being characterized by one arbitrary constant.

From (3.1), (1.1) and (1.4), it is easy to see that the generator $A_1A_4\in\Phi_1$ is fixed and A_2 moves on the absolute σ. This complex consists of all bundles of lines with vertices on σ and A_1A_4 as a layer.

Analogous to the above investigation, we have three classes of line complexes $HGr_3(1,3,3)$ defined as follows:

The complete integrable Pfaffian equation

$$\omega_1^4=0, \tag{3.2}$$

determines a line complex. This line complex constructed geometrically as the set of all bundles of lines with vertices on the absolute σ and $A_1A_3\in\Phi_2$ as a layer.

The involutive equation

$$\omega_2^4=0, \tag{3.3}$$

characterizes a line complex which consists of the family of all bundles of lines with vertices on the absolute σ and $A_2A_3\in\Phi_1$ as a layer.

The differential equation

$$\omega_2^3=0;\ D\omega_2^3=0, \tag{3.4}$$

defines a line complex. This line complex represented as the set of all bundles with layer $A_2A_4\in\Phi_2$ and vertices on the absolute σ. From the foregoing results, we have the following lemmas.

LEMMA 3.1. *The intersection of the line complexes* $HGr_3(1,3,3)$ (3.1), (3.3) *determines a hyperbolic linear congruences* [8].

$$\omega_1^3=0,\ \omega_2^4=0, \tag{3.5}$$

with two directrices belonging to the family Φ_1.

LEMMA 3.2. *The set of lines common to the line complexes* $HGr(1,3,3)$ (3.2), (3.4), *that is, the set of lines satisfy the integrable system of equations.*

$$\omega_1^4=0,\ \omega_2^3=0, \tag{3.6}$$

determines a hyperbolic linear congruence with two directrices belonging to the family Φ_2.

Therein, we give the parametric equations of one class of $HGr_3(1,3,3)$ [9]. Say the class of line complexes (3.1). One way of obtaining an integral-free representation of such complexes is the following. We take a generator l of the ruled absolute σ, for each point on the absolute σ draw a bundle of lines with l as a layer. All these bundles construct the class of line complexes (3.1). Using this representation, we shall find the equations of the complexes (3.1) in Plücker coordinates as follows: Consider two points $P_1(\lambda,1,1,\lambda)$, $P_2(1,\lambda,1/\lambda,\lambda^2)$ on the generator l of the absolute σ. Also any arbitrary point on σ is $Q(1,\nu,\nu^2,1/\nu)$. The Plücker coordinates of the line PQ which is a ray of the line complex $\{P=P_1+tP_2$ is a point on $l\}$ are

$$\begin{aligned}
&P^{12}=\nu(\lambda+t)-(1+\lambda t),\ P^{13}=(\lambda+t)(\nu^2-1/\lambda)\\
&P^{14}=(\lambda+t)/\nu-\lambda(1+\lambda t),\ P^{23}=\nu^2(1+\lambda t)-\nu(1+t/\lambda)\\
&P^{24}=(1+\lambda t)(1/\nu-\nu\lambda),\ P^{34}=(\lambda+t)/(\lambda\nu)-\lambda\nu^2(1+\lambda t)
\end{aligned} \tag{3.7}$$

These equations depends on three parameters (λ, t, ν), they represent the parametric equations of the constructed line complex (3.1).

4. Class of the seminonholonomic line complexes $SNGr_2(1,3,3;h)$

We consider the special class of line complexes $NGr(1,3,3)$ in which the rank of the matrix M equal to one. This class is classified in three separate subclasses. The matrix M has rank equal to one if and only if one of the following conditions

(I) $a=b=0$, (II) $a=\chi=0$, (III) $b=\chi=0$, is satisfied. We denote by $NGr_2(1,3,3)$ the class of line complexes under investigation.

DEFINITION 4.1. [5] Let a field of a differential-geometric object θ having the same structure as the subobject $h\subset\theta$ be given on a line complex $NGr(1,3,3)$.

Then we say that $NGr_2(1,3,3)$ is a seminonholonomic line complex $SNGr_2(1, 3, 3; h)$.

Three subclasses of $NGr_2(1,3,3)$ are examined: The line complexes $SNGr_2(1,3,3; \{\chi\})$, $SNGr_2(1, 3, 3; \{b\})$ and $SNGr_2(1, 3, 3; \{a\})$ according to the conditions (I), (II) and (III) respectively.

The line complexes $SNGr_2(1, 3, 3; \{x\})$ is determined by the differential equation.

$$\omega_1^3=\chi\omega_2^4 \tag{4.1}$$

and the associated exterior quadratic equation. This is an involutive equation, the nonuniqueness of its solutions being characterized by one arbitrary function of one argument.

Since the normal correspondences $K: A_1 \longleftrightarrow \Sigma(A_1): x^3=0$, $K: A_2 \longleftrightarrow \Sigma(A_2): x^4=0$, are established for the complex (4.1), the points A_1, A_2 are called the *centres* of the ray l [10]. This gives a geometric interpretation of the subobject $h=\{\chi\}\subset\theta$, χ is called the *curvature of the line complex* (4.1). Since the equation $\omega_2^4=0$, is complete integrable with respect to the line complex (4.1), it is easy to see that, this equation defines a holonomic line congruence coincident with the linear line congruence (3.5).

LEMMA 4.1. *The line complex* (4.1) *admits a stratification into one-parameter families of hyperbolic linear line congruences* (3.5).

The line complexes $SNGr_2(1,3,3; \{b\})$ is defined by the involutive system of differential equation

$$\left.\begin{aligned} &\omega_1^3=b\,\omega_1^4 \\ &\{db+2b\omega_3^3\}\wedge\omega_1^4=0 \end{aligned}\right] \tag{4.2}$$

The range of existence of such line complex comprises one arbitrary function of one argument. The first fundamental differential-geometric subobject $h=\{b\}\subset\theta$ is established by the fixed correspondence

$$K: M(t)=A_1+tA_2 \longleftrightarrow \Sigma(M(t)): bx^3+x^4=0.$$

As the point M ranges over the ray l, the plane $\Sigma(M)$ is fixed, i.e., the cone of rays of the line complex (4.2), passing through M is degenerate into a cylinderical surface.

The system of complete integrable Pfaffian equations

$$\omega_1^3=0,\ \omega_1^4=0 \tag{4.3}$$

determines a parabolic holonomic line congruence {with focal surface degenerate into the point A_1} belonges to the line complex (4.2).

LEMMA 4.2. *The line complex* (4.2) *admits a fibration into one-parameter families of parabolic line congruences* (4.3).

The line complexes $SNGr_2(1, 3, 3; \{a\})$ is characterized by the integrable system of equations

$$\left.\begin{array}{l}\omega_1^3=a\omega_2^3\\ \left\{da-2a\omega_1^1\right\}\wedge\omega_2^3=0\end{array}\right] \tag{4.4}$$

This system exists within one arbitrary function of one argument. The geometric interpretation of the subobject $h=\{a\}\subset\theta$ follows from the fixed correspondence $K: M(t)=A_1+tA_2,\quad a+t\neq 0 \longleftrightarrow \Sigma(M(t))x^4=0.$ The complete integrable Pfaffian system of equations

$$\omega_1^3=0,\ \omega_2^3=0,\ \omega_1^4=0 \tag{4.5}$$

determines a ruled surface which is called the ***integral ruled surface*** of the line complex (4.4). This ruled surface degenerate into a pencil of lines with centre at A_1 in the fixed plane $A_1A_2A_4$.

LEMMA 4.3. *The line complex* (4.4) *can be represented as the set of two-parameter families of the penciles of straight lines* (4.5).

Faculty of Science
Assiut Univ.
Assiut, Egypt

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