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GENERALIZED LOGARITHMIC MEANS OF AN ENTIRE DIRICHLET SERIES OF ORDER ZERO (I)

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1. Let $f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $s=\sigma+it$, $\lambda_{n+1}>\lambda_n$, $\lambda_1\geqslant 0$, $\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n}=0$, represent an entire function. The Ritt order $\rho(0\leqslant\rho\leqslant\infty)$ of $f(s)$ is defined [4, *p.* 78] as the limit superior of $\left\{\frac{\log \log M(\sigma)}{\sigma}\right\}$, as $\sigma\to\infty$, with $M(\sigma)=\sup\{|f(\sigma+it)| : -\infty<t<\infty\}$. For a class of entire functions of Ritt order zero, we define L-order (logarithmic order in the sense of Reddy) ρ^* and lower L-order λ^* as [3]:

$$(1.1) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log M(\sigma)}{\log \sigma} = {}^{\rho^*}_{\lambda^*}, \quad 1\leqslant\lambda^*\leqslant\rho^*\leqslant\infty.$$

For functions of L-order ρ^* $(1\leqslant\rho^*\leqslant\infty)$, the L-type T^* and lower L-type t^* are defined as:

$$(1.2) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{\sigma^{\rho^*}} = {}^{T^*}_{t^*}, \quad 0\leqslant t^*\leqslant T^*\leqslant\infty.$$

The logarithmic mean of $f(s)$ is defined [1, p. 13] as:

$$(1.3) \qquad L(\sigma)=\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T} \log|f(\sigma+it)|\,dt\right\}.$$

Also, for $\delta>0$, we consider the following generalized logarithmic means of $f(s)$:

$$(1.4) \qquad L_{\delta}(\sigma)=e^{-\delta\sigma}\int_0^{\sigma} e^{\delta x}L(x)\,dx,$$

and

$$(1.5) \qquad L^*_{\delta}(\sigma)=\sigma^{-\delta-1}\int_0^{\sigma} x^{\delta}L(x)\,dx.$$

Our aim in this paper is to study some of the growth properties of the logarithmic mean $L(\sigma)$ and the generalized logarithmic mean $L^*_{\delta}(\sigma)$. Various constants have been defined and a number of relations involving these constants have been obtained.

2. Since $\log L^*_\delta(\sigma)$ is an increasing convex function of $\log\sigma$ [6, Lemma 2], we may write

$$(2.1)\qquad \log L^*_\delta(\sigma)=\log L^*_\delta(\sigma_0)+\int_{\sigma_0}^{\sigma}\frac{U(x)}{x}dx,\quad \sigma>\sigma_0,$$

where $U(x)$ is a positive real valued indefinitely increasing function of x [2, p. 73]. For $1<\rho^*<\infty$, let us set,

$$(2.2)\qquad \lim_{\sigma\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\frac{U(\sigma)}{\sigma^{\rho^*}}=\begin{matrix}\alpha\\ \beta\end{matrix},\quad 0<\beta\leqslant\alpha<\infty,$$

and

$$(2.3)\qquad \lim_{\sigma\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\frac{\log L^*\delta(\sigma)}{\sigma^{\rho^*}}=\begin{matrix}p\\ q\end{matrix},\quad 0<q\leqslant p<\infty.$$

We prove the following:

THEOREM 1. *If $L^*_\delta(\sigma)$ is the generalized logarithmic mean of $f(s)$ of L-order ρ^* $(1<\rho^*<\infty)$ and α,β; p,q are the constants defined as in* (2.2) *and* (2.3), *then*

$$(2.4)\qquad \beta\leqslant\rho^*\ q\leqslant\rho^*\ p\leqslant\alpha,$$

$$(2.5)\qquad \rho^*\ P\geqslant\frac{\alpha}{e}\ e^{\beta/\alpha}\geqslant\beta,$$

$$(2.6)\qquad \rho^*\ q\leqslant\beta\left\{1+\log\frac{\alpha}{\beta}\right\}\leqslant\alpha,$$

and

$$(2.7)\qquad \alpha+\rho^*\ q\leqslant e\ \rho^*\ p.$$

PROOF. From (2.2), we have, for any $\epsilon>0$ and $\sigma\geqslant\sigma_0$,

$$(2.8)\qquad (\beta-\epsilon)\sigma^{\rho^*}<U(\sigma)<(\alpha+\epsilon)\sigma^{\rho^*}.$$

Also for $h>0$, we have

$$(2.9)\qquad \log L^*_\delta(\sigma+h\sigma)=\log L^*_\delta(\sigma_0)+\left\{\int_{\sigma_0}^{\sigma}+\int_{\sigma}^{\sigma+h\sigma}\right\}\frac{U(x)}{x}dx.$$

Using left-hand inequality of (2.8) in (2.9), we get

$$\log L^*_\delta(\sigma+h\sigma)>0(1)+(\beta-\epsilon)\int_{\sigma_0}^{\sigma}x^{\rho^*-1}dx+U(\sigma)\int_{\sigma}^{\sigma+h\sigma}\frac{dx}{x}$$
$$=0(1)+(\beta-\epsilon)(\rho^*)^{-1}\sigma^{\rho^*}(1-0(1))+U(\sigma)\log(1+h).$$

So

$$\frac{\log L^*_\delta(\sigma+h\sigma)}{(\sigma+h\sigma)^{\rho^*}}>\frac{1}{(1+h)^{\rho^*}}\left\{0(1)+\frac{\beta-\epsilon}{\rho^*}(1-0(1))+\frac{U(\sigma)}{\sigma^{\rho^*}}\log(1+h)\right\}.$$

Taking limits, as $\sigma \to \infty$, we get

$$p \geqslant (1+h)^{-\rho^*}\{\beta(\rho^*)^{-1}+\alpha\log(1+h)\}, \tag{2.10}$$

and

$$q \geqslant (1+h)^{-\rho^*}\{\beta(\rho^*)^{-1}+\beta\log(1+h)\}. \tag{2.11}$$

Similarly, on using right-hand inequality of (2.8) in (2.9), we get

$$p \leqslant \alpha(\rho^*)^{-1}(1+h)^{-\rho^*}+\alpha\log(1+h), \tag{2.12}$$

and

$$q \leqslant \alpha(\rho^*)^{-1}(1+h)^{-\rho^*}+\beta\log(1+h). \tag{2.13}$$

It can be seen after a long calculation that the maxima of the right-hand side expressions in (2.10) and (2.11) occur at

$$h = e^{(\alpha-\beta)/\rho^*\alpha}-1, \tag{2.14}$$

and

$$h = 0, \tag{2.15}$$

respectively. Substituting these values of h from (2.14) and (2.15) in (2.10) and (2.11), respectively, we get

$$\rho^*\ p \geqslant \frac{\alpha}{e}e^{\beta/\alpha} \geqslant \beta, \tag{2.16}$$

and

$$\rho^*\ q \geqslant \beta, \tag{2.17}$$

the last inequality in (2.16) follows from the fact that

$$e^x \geqslant ex \text{ for } x \geqslant 0.$$

It can also be seen that the minima of the right-hand side expressions in (2.12) and (2.13) occur at

$$h = 0, \tag{2.18}$$

and

$$h = (\alpha/\beta)^{1/\rho^*}-1, \tag{2.19}$$

respectively. Substituting these values of h from (2.18) and (2.19) in (2.12) and (2.13), respectively, we get

$$\rho^*\ p \leqslant \alpha, \tag{2.20}$$

and

$$\rho^*\ q \leqslant \beta\ (1+\log(\alpha/\beta)) \leqslant \alpha, \tag{2.21}$$

since $1+\log x \leqslant \exp(\log x)$.

Combining (2.17) and (2.20), since $\rho^*\ q \leqslant \rho^*\ p$, we get (2.4), (2.5) and (2.6) follow from (2.16) and (2.21), respectively.

To prove (2.7), we note that

$$U(\sigma)\leqslant\rho^*\int_{\sigma}^{e^{1/\rho^*}\sigma}\frac{U(x)}{x}dx.$$

Adding $\rho^*\log L^*_{\delta}(\sigma)$ on both sides of the above relation and using (2.1), we get

$$\rho^*\log L^*_{\delta}(\sigma)+U(\sigma)\leqslant\rho^*\log L^*_{\delta}(e^{1/\rho^*}\sigma).$$

Dividing throughout by σ^{ρ^*}, proceeding to limits and using (2.2) and (2.3), we get (2.7). This completes the proof of the Theorem 1.

REMARK. In the relation (2.5) actually $\beta<\frac{\alpha}{e}e^{\beta/\alpha}$ if $\alpha\neq\beta$, and in relation (2.6) $\beta(1+\log(\alpha/\beta))<\alpha$ if $\alpha\neq\beta$. Thus the equality in the relation (2.5) and (2.6) will occur only if $\alpha=\beta$. Moreover, from (2.5), $\frac{\alpha}{e}e^{\beta/\alpha}\leqslant\rho^* p$, or

$$\alpha+\beta\leqslant e\,\rho^* p. \tag{2.22}$$

A comparison of (2.7) and (2.22) shows that (2.7) is a refinement of (2.22).

THEOREM 2. *If $f(s)$ is an entire function of L-order $\rho^*(1<\rho^*<\infty)$, then $\lim_{\sigma\to\infty}\frac{U(\sigma)}{\sigma^{\rho^*}}$ exists, if and only if, $\lim_{\sigma\to\infty}\frac{\log L^*_{\sigma}(\sigma)}{\sigma^{\rho^*}}$ exists, in which case*

$$\lim_{\sigma\to\infty}\frac{U(\sigma)}{\sigma^{\rho^*}}=\rho^*\lim_{\sigma\to\infty}\frac{\log L^*_{\delta}(\sigma)}{\sigma^{\rho^*}}. \tag{2.23}$$

PROOF. If $\lim_{\sigma\to\infty}\frac{U(\sigma)}{\sigma^{\rho^*}}$ exists, then it follows that $\lim_{\sigma\to\infty}\frac{\log L^*_{\delta}(\sigma)}{\sigma^{\rho^*}}$ exists from (2.4). We, therefore, suppose that

$$\lim_{\sigma\to\infty}\frac{\log L^*_{\delta}(\sigma)}{\sigma^{\rho^*}}=p. \tag{2.24}$$

First let, $0<p<\infty$, then for any $\epsilon>0$ and $\sigma\geqslant\sigma_0$, we have

$$(p-\epsilon)\sigma^{\rho^*}<\log L^*_{\delta}(\sigma)<(p+\epsilon)\sigma^{\rho^*}. \tag{2.25}$$

Hence, for any $h>0$, we get

$$\int_{\sigma}^{\sigma+h\sigma}\frac{U(x)}{x}dx=\int_{0}^{\sigma+h\sigma}\frac{U(x)}{x}dx-\int_{0}^{\sigma}\frac{U(x)}{x}dx$$
$$=\log L^*_{\delta}(\sigma+h\sigma)-\log L^*_{\delta}(\sigma)$$

$$<(p+\epsilon)(\sigma+h\sigma)^{\rho^*}-(p-\epsilon)\sigma^{\rho^*}$$
$$=p(\rho^*h+\dots)\sigma^{\rho^*}+\epsilon(2+\rho^*h+\dots.)\sigma^{\rho^*}.$$

Therefore

$$U(\sigma)\log(1+h)\leqslant\int_{\sigma}^{\sigma+h\sigma}\frac{U(x)}{x}\,dx<p(p^*h+\dots)\sigma^{\rho^*}+\epsilon(2+\rho^*h+\dots)\sigma^{\rho^*}.$$

Since ϵ and h are arbitrary and so making $\epsilon\to0$ and $h\to0$, we have

$$\limsup_{\sigma\to\infty}\frac{U(\sigma)}{\sigma^{\rho^*}}\leqslant\rho^*p. \tag{2.26}$$

Proceeding in a similar way, we easily get

$$U(\sigma)\log\left\{\frac{1}{1-h}\right\}\geqslant\int_{\sigma-h\sigma}^{\sigma}\frac{U(x)}{x}\,dx>p(\rho^*h-\dots)\sigma^{\rho^*}-\epsilon(2-\rho^*h+\dots)\sigma^{\rho^*}.$$

This gives

$$\liminf_{\sigma\to\infty}\frac{U(\sigma)}{\sigma^{\rho^*}}\geqslant\rho^*p. \tag{2.27}$$

Thus, for $0<p<\infty$, the relations (2.26) and (2.27) give us

$$\lim_{\sigma\to\infty}\frac{U(x)}{\sigma^{\rho^*}}=\rho^*p. \tag{2.28}$$

If $p=0$, then (2.26) gives $\lim_{\sigma\to\infty}\frac{U(\sigma)}{\sigma^{\rho^*}}=0$ and if $p=\infty$, then taking an arbitrarily large number M in place of $p-\epsilon$ and proceeding as above, we get $\lim_{\sigma\to\infty}\frac{U(\sigma)}{\sigma^{\rho^*}}=\infty$.

The equality (2.23) follows from (2.24) and (2.28).

THEOREM 3. *For every entire function $f(s)$ of L-order $\rho^*(1\leqslant\rho^*\leqslant\infty)$ and lower L-order λ^* $(1\leqslant\lambda^*\leqslant\infty)$,*

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{L(\sigma)}{L^*_{\delta}(\sigma)}-(\delta+1)\right\}\leqslant\begin{matrix}2\rho^*\\ 2\lambda^*\end{matrix}, \tag{2.29}$$

In order to prove this theorem we need the following lemmas:

LEMMA 1. *$\sigma^{\delta+1}L(\sigma)$ is an increasing convex function of $\sigma^{\delta+1}L^*_{\delta}(\sigma)$.*

LEMMA 2. *For $\Delta>\sigma$, we have*

$$L^*_{\delta}(\sigma)\leqslant\frac{L(\sigma)}{\delta+1}\leqslant\frac{\Delta^{\delta+1}}{\Delta^{\delta+1}-\sigma^{\delta+1}}L^*_{\delta}(\Delta). \tag{2.30}$$

LEMMA 3. *We have*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log L(\sigma)}{\log \sigma} \leqslant \begin{matrix}\rho^*\\ \lambda^*\end{matrix}. \tag{2.31}$$

The proofs of the above three lemmas are given by the author ([5], [6]).

LEMMA 4. *We find*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log L^*_\delta(\sigma)}{\log \sigma} \leqslant \begin{matrix}\rho^*\\ \lambda^*\end{matrix}. \tag{2.32}$$

This lemma follows in view of (2.30) and (2.31).

PROOF OF THEOREM 3. It is readily seen that

$$\frac{d}{dx}\Big\{\{(\delta+1)\log x + \log L^*_\delta(x)\Big\} = \frac{1}{x}\Big\{\frac{L(x)}{L^*_\delta(x)}\Big\},$$

so that

$$(\delta+1)\log\frac{\sigma}{\sigma_0} + \log L^*_\delta(\sigma) - \log L^*_\delta(\sigma_0) = \int_{\sigma_0}^{\sigma} \frac{L(x)}{L^*_\delta(x)}\ \frac{dx}{x},$$

that is

$$\log L^*_\delta(\sigma) = \log L^*_\delta(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{m(x)}{x} dx, \tag{2.33}$$

where

$$m(x) = \Big\{\frac{L(x)}{L^*_\delta(x)} - (\delta+1)\Big\} \tag{2.34}$$

increases with x, by virtue of Lemma 1. Thus, for $\sigma \geqslant \sigma_0$, (2.33) gives

$$\log L^*_\delta(\sigma^2) \geqslant \int_{\sigma}^{\sigma^2} \frac{m(x)}{x} dx \geqslant m(\sigma)\log\sigma.$$

Hence,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\ m(\sigma) \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{2\log L^*_\delta(\sigma^2)}{\log \sigma^2},$$

which gives the desired result in view of (2.32) and (2.34).

THEOREM 4. *If*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log L^*_\delta(\sigma)}{\log \sigma} = \begin{matrix}A\\ B\end{matrix},\quad 0 \leqslant B \leqslant A \leqslant \infty,$$

then

$$\lim_{\sigma\to\infty}\inf\frac{\log L^*_\delta(\sigma)}{U(\sigma)}\leqslant\frac{1}{A}\leqslant\frac{1}{B}\leqslant\lim_{\sigma\to\infty}\sup\frac{\log L^*_\delta(\sigma)}{U(\sigma)},$$

where $U(\sigma)$ is given by (2.1).

The proof of this theorem follows on the lines of Theorem 3 obtained by the author [5].

Finally, we have

THEOREM 5. *If $L^*_\delta(\sigma)$ is the generalized logarithmic mean of $f(s)$ of L-order ρ^* $(1<\rho^*<\infty)$, L-type T^* and lower L-type t^*, then*

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L^*_\delta(\sigma)}{\sigma^{\rho^*}}\leqslant\begin{matrix}T^*/(\delta+1)\\ t^*/(\delta+1)\end{matrix}.$$

PROOF. From (1.3) and (1.5), we have

$$L^*_\delta(\sigma)=\lim_{T\to\infty}\left\{\frac{1}{2T\sigma^{\delta+1}}\int_0^\sigma\int_{-T}^{T}x^\delta\log|f(x+it)|\,dx\;dt\right\}$$
$$\leqslant\frac{\log M(\sigma)}{\delta+1}.$$

Therefore

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L^*_\delta(\sigma)}{\sigma^{\rho^*}}\leqslant\left(\frac{1}{\delta+1}\right)\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log M(\sigma)}{\sigma^{\rho^*}}.$$

Hence the result follows in view of (1.2).

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