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A REMARK ON THE CLASSES $D(k)$ AND $R(k)$

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1. Introduction

Let S_0 denote the class of functions

$$f(z)=\sum_{n=1}^{\infty} a_n z^n$$

that are analytic and univalent in the unit disk $U=\{|z|<1\}$. For $k(0\leqq k<1)$, let $S_0^*(k)$ and $K_0(k)$ denote the subclasses of S_0 consisting of the functions that are starlike of order k and convex of order k, respectively.

For these classes $S_0^*(k)$ and $K_0(k)$, H. Silverman [11] showed the following lemmas.

LEMMA 1. *A function*

$$f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n\geqq 0)$$

is in the class $K_0(k)$ *if and only if*

$$\sum_{n=2}^{\infty} n(n-k)a_n \leqq a_1(1-k).$$

LEMMA 2. *A function*

$$f(z)=a_1 z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n\geqq 0)$$

is in the class $S_0^*(k)$ *if and only if*

$$\sum_{n=2}^{\infty} (n-k)a_n \leqq a_1(1-k).$$

Let $S_f(z)$ be the Schwarzian derivative of $f(z)$ at $z\in U$, that is,

$$S_f(z)=\left(\frac{f''(z)}{f'(z)}\right)'-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

And let $\rho(z)$ be the density of non-euclidean metric defined in the unit disk U. Then, R. Kuehnau [1] gave the following lemma for the Schwarzian derivative.

LEMMA 3. *If the function $f(z)$ is analytic and univalent in the unit disk U, then*

$$\sup_{z\in U} U|Sf(z)|\rho(z)^{-2}\leqq 6.$$

2. Some results for the classes $D(k)$ and $R(k)$

DEFINITION 1. Let $D(k)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk U and satisfying

$$\left|\frac{f'(z)-1}{f'(z)+1}\right|<k \qquad (z\in U)$$

for some k $(0<k\leqq 1)$.

For this class, K.S. Padmanabhan [7] gave the following lemmas.

LEMMA 4. *If the function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

belongs to the class $D(k)$, then we have

$$|f'(z)|\leqq\frac{1+k|z|}{1-k|z|},$$

$$\mathrm{Re}f'(z)\geqq\sqrt{\frac{1-k|z|}{1+k|z|}},$$

$$|f(z)|\geqq-|z|+\frac{2}{k}\log(1+k|z|),$$

and

$$|f(z)|\leqq-|z|-\frac{1}{k}\log(1-k|z|)$$

for $z\in U$.

LEMMA 5. *Let the function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

belongs to the class $D(k)$. Then, for any $n\geqq 2$,

$$|a_n|\leqq\frac{2k}{n}.$$

REMARK 1. In 1980, S. Owa [6] showed some results for the fractional calculus of functions $f(z)$ in the class $D(k)$.

DEFINITION 2. Let $R(k)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

analytic in the unit disk U and satisfying

$$\operatorname{Re}f'(z)>k$$

for some k $(0\leqq k<1)$.

For this class, D. B. Shaffer [10] showed the following lemma.

LEMMA 6. *Let the function*

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

be in the class $R(k)$. Then,

$$\left|\frac{f''(z)}{f'(z)}\right|\leqq\frac{2(1-k)}{(1-|z|)\{1+(1-2k)|z|\}}$$

for $z\in U$.

THEOREM 1. *If the function*

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

belongs to the class $D(k)$, then the function $f(z)$ is in the class $R\{(1-k)/(1+k)\}$, that is, the class $D(k)$ is a subclass of $R\{(1-k)/(1+k)\}$.

PROOF. Since the function $f(z)$ is in the class $D(k)$, by using Lemma 4, we have

$$\begin{aligned}\operatorname{Re}f'(z)&\geqq\frac{1-k|z|}{1+k|z|}\\&>\frac{1-k}{1+k}\end{aligned}$$

for $z\in U$. Furthermore,

$$0\leqq\frac{1-k}{1+k}<1,$$

for $0<k\leqq 1$. This completes the proof of the theorem.

COROLLARY 1. *In particular, if the function $f(z)$ is in the class $D(\sqrt{2}-1)$, then, $f(z)$ belongs to the class $R(\sqrt{2},1)$, and if the function $f(z)$ is in the class $D(1)$, then, $f(z)$ belongs to the class $R(0)$.*

THEOREM 2. *Let the function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n\geqq 0)$$

belong to the class $S_0^*(0)$*. Then, the function* $f(z)$ *is in the class* $D(1)$*.*

PROOF. Since $f(z)\in S_0^*(0)$, by using Lemma 2, we have

$$\left|\frac{f''(z)-1}{f'(z)+1}\right| \leqq \frac{\sum_{n=2}^{\infty} na_n|z|^{n-1}}{2-\sum_{n=2}^{\infty} na_n|z|^{n-1}}$$

$$\leqq \frac{|z|\sum_{n=2}^{\infty} na_n}{2-|z|\sum_{n=2}^{\infty} na_n}$$

$$\leqq \frac{|z|}{2-|z|}$$

$$<1.$$

Hence, the function $f(z)$ belongs to the class $D(1)$.

THEOREM 3. *Let the function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \quad (a_n \geqq 0)$$

be in the class $K_0(k)$*. Then, the function* $f(z)$ *belongs to the class* $D\{(1-k)/(3-k)\}$*.*

PROOF. Since the function $f(z)$ is in the class $K_0(k)$, by using Lemma 1,

$$(2-k)\sum_{n=2}^{\infty} na_n \leqq \sum_{n=2}^{\infty} n(n-k)a_n$$

$$\leqq 1-k$$

for $0\leqq k<1$. Hence, we have

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| \leqq \frac{|z|\sum_{n=2}^{\infty} na_n}{2-|z|\sum_{n=2}^{\infty} na_n}$$

$$<\frac{1-k}{3-k}$$

$$<1.$$

Therefore, the function $f(z)$ is in the class $D\{(1-k)/(3-k)\}$.

COROLLARY 2. *Let the function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \quad (a_n \geqq 0)$$

be in the class $K_0(0)$. *Then, the function* $f(z)$ *belongs to the class* $D(1/3)$.

COROLLARY 3. *Under the hypotheses of Theorem* 3,

$$a_n \leqq \frac{2(1-k)}{n(3-k)}$$

for $0 \leqq k < 1$ *and any* $n \geqq 2$.

THEOREM 4. *Let the function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_0$$

belong to the class $D(k)$. *Then, we have*

$$|f'''(z)| \leqq \frac{6(1+k|z|)}{(1-|z|)^2(1-k|z|)} \left\{ \frac{1}{(1+|z|)^2} + \frac{4k^2}{\{1+k+(3k-1)|z|\}^2} \right\}.$$

PROOF. By using Lemma 3 and Lemma 6,

$$\left| \frac{f'''(z)}{f'(z)} \right| \leqq 6\rho(z)^2 + \frac{24k^2}{(1-|z|)^2 \{1+k+(3k-1)|z|\}^2}.$$

Hence, we have the theorem with the aid of Lemma 4.

3. An application for the fractional derivative

There are many definitions of the fractional derivative. In 1978, S. Owa [6] gave the following definitions for the fractional derivative of order α.

DEFINITION 3. The fractional derivative of order α is defined by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^\alpha},$$

where $0 < \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Moreover,

$$f(z) = \lim_{\alpha \to 0} D_z^\alpha f(z)$$

and

$$f'(z) = \lim_{\alpha \to 1} D_z^\alpha f(z).$$

DEFINITION 4. Under the hypotheses of Definition 1, the fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n} D_z^\alpha f(z),$$

where $n \in N \cup \{0\}$.

REMARK 2. For other definitions of the fractional derivative of order α, see [2], [4], [8], [3] and [9].

THEOREM 5. *Let the function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \quad (a_n \geqq 0)$$

belong to the class $K_0(0)$. *Then, for* $0<\alpha<1$ *and* $z\in U$,

$$|D_z^{\alpha} f(z)| \geqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\{-|z|+2\log(1+|z|)\},$$

$$|D_z^{\alpha} f(z)| \leqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\{-|z|-2\log(1-|z|)\}$$

and

$$|D_z^{1+\alpha} f(z)| \leqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\left\{\frac{1+|z|}{1-|z|}-\alpha-\frac{2\alpha}{|z|}\log(1-|z|)\right\}.$$

PROOF. Let consider the function

$$G(z)=\Gamma(2-\alpha)z^{\alpha} D_z^{\alpha} f(z).$$

Then, by using Lemma 1, we have

$$\left|\frac{G'(z)-1}{G'(z)+1}\right| \leqq \frac{\sum_{n=2}^{\infty} \frac{n\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n |z|^{n-1}}{2-\sum_{n=2}^{\infty} \frac{n\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n |z|^{n-1}}$$

$$\leqq \frac{|z|\sum_{n=2}^{\infty} n^2 a_n}{2-|z|\sum_{n=2}^{\infty} n^2 a_n}$$

$$\leqq \frac{|z|}{2-|z|}$$

$$<1.$$

Therefore, the function $G(z)$ belongs to the class $D(1)$. By using Lemma 4, we have

$$|G(z)| \geqq -|z|+2\log(1+|z|),$$

$$|G(z)| \leqq -|z|-2\log(1-|z|)$$

and

$$|G'(z)| \leqq \frac{1+|z|}{1-|z|}.$$

And three estimates we desire follow these inequalities.

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REFERENCES

[1] R. Kuehnau, ***Zur abschaetzung der Schwarzschen ableitung bei schlichten funktionen***, Math. Nachr., 59(1974), 195—198.

[2] J. Liouville, ***Mémoire sur le calcul de différentielles à indices quelconques***, J. École Polytech., 13(1832), 71—162.

[3] K. Nishimoto, ***Fractional derivative and integral* I**, J. Coll. Engrg. Nihon Univ., 17(1976), 11—19.

[4] T. J. Osler, *Leibniz rule for fractional derivative generalized and application to infinite series*, Siam J. Appl. Math., 16(1970), 658—674.

[5] S. Owa, *On the distortion theorems I*, Kyungpook Math. J., 18(1978), 53—59.

[6] ______, *On applications of the fractional calculus*, Math, Japon., 25(1980), 195—206.

[7] K. S. Padmanabhan, *A certain class of functions whose derivatives have a positive real part in the unit disk*, Ann. Polon. Math., 23(1970), 73—81.

[8] B. Ross, *The development, theory and applications of the Gamma-function and a profile of fractional-calculus*, Thesis, New York Univ., 1974.

[9] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ., 11(1978), 135—143.

[10] D. B. Shaffer, *On bounds for the derivative of analytic functions*, Proc. Amer. Math. Soc., 37(1973), 517—520.

[11] H. Silverman, *Extreme points of univalent functions with two fixed points*, Trans. Amer. Math. Soc., 219(1976), 387—395.