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# D.G. NEAR RINGS ON THE DIHEDRAL GROUP OF ORDER $2n$, $n$ EVEN

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## 1. Introduction

Previous work on d.g. near rings associated with the dihedral groups includes the papers [1] through [5] and the thesis [7]. In [3] it is shown that the number of nonisomorphic d.g. near rings definable on $D_{2n}$, $n$ odd, is $1+2^r$, where $r$ is the number of distinct primes occurring in the factorization of $n$. Pilz [6] summarizes some of the results given in the other references, and may also be consulted for definitions and basic results of near rings. The results in the next two lemmas are taken, respectively, from [2] and [7].

LEMMA 1. *Let $(G, +)$ be a group and let $K$ be an additive generating set for $G$ with an associative multiplication defined on it. If the multiplication on $K$ may be extended so that, first, each element of $K$ is right distributive and, secondly, each element of $G$ is left distributive, then $(G, \cdot)$ is associative. Thus $(G, +, \cdot)$ is a d.g. near ring with generating set $K$.*

LEMMA 2. *If $(D_8, +, \cdot)$ is a d.g. near ring with trivial left annihilator, then $(D_8, +, \cdot)$ is the unique d.g. near ring with identity definable on $D_8$.*

The dihedral group of order $2n$ will be designated by $D_{2n}$ and presented as $(a, b \mid na,\ 2b,\ a+b+a+b)$. Elements of $D_{2n}$ will be given in the form $xa+zb$, $0\leq x\leq n-1$, $0\leq z\leq 1$. For the remainder of this paper it is assumed that $n$ is even and $n\geq 4$. If $N$ is a left distributive d.g. near ring defined on $(D_{2n}, +)$, $K$ will stand for its generating set (in the d.g. sense). Also, $Z_n$ will designate the additive cyclic group of order $n$.

In [4] it was shown that the proper normal subgroups of $D_{2n}$ are the subgroups of (a), the normal subgroup $S$ generated by $b$, and the normal subgroup $T$ generated by $a+b$. $S$ and $T$ have index 2.

The symbol $2^k \| n$ is used to indicate $2^k \mid n$ but $2^{k+1} \nmid n$.

## 2. Main result

THEOREM. *If $2 \| n$ or if $2^3 \mid n$ there are exactly 19 non-isomorphic d.g. near rings* *which can be defined on* $D_{2n}$. *If* $2^2 \| n$, *there are exactly* 20. *In the latter case, the additional near ring* $N$ *is such that* $N/L$ *is isomorphic to the unique d.g. near ring with identity on* $D_8$ *where* $L=(4a)$ *is the ideal of left annihilators of* $N$.

PROOF. First consider the case in which $K$ can be taken so as to contain no element with order greater than 2. An element of order 2 can only have elements of order 1 or 2 in its column of the multiplication table since these column entries are images of an element of order 2 under the various row endomorphisms. If an element of order 2 is right distributive, it defines an endomorphism of $D_{2n}$. Since each element of the column has order at most 2, this column endomorphic image is abelian. Thus the column image for an element of $K$ is a $Z_1$; a $Z_2$; or a $D_4$ (Klein group) and the column kernel is $D_{2n}$; $(a)$, $S$, or $T$; or $(2a)$. In any case, $2a$ is in the column endomorphism kernel of a generator so that $(2a) \leq L$. Thus $L$ has index at most 4 and at most 4 distinct row endomorphisms occur in the multiplication table. The arguments made in [7] for $n=4$ cover all cases in which $(2a) \leq L$ and readily extend to arbitrary even $n$. The 19 non-isomorphic near rings which result are described in Table 1. An abbreviated multiplication table for $K$ is given for each d.g. near ring to save space the rows and columns of 0 and some elements of order 2 are not given and $(n/2)a$ is indicated by the symbol $a$. Note that near rings 1 through 16 are distributive.

We now turn to the main argument which concerns the exceptional cases. Consider the case in which $K$ must contain an element whose order is greater than 2 and call this element $ya$. Obviously, $K$ must also contain at least one element of the form $ta+b$. Since the column kernel of $ta+b$ must contain $(2a)$, the row kernel of each element of the form $2ra$ contains $ta+b$. The normal subgroup generated by $ta+b$ is $S$ if $t$ is even and $T$ if $t$ is odd. Thus, for each $r$, the row image of $2ra$ has order at most 2. In particular, $(2a)(ya)$ has order 1 or 2 and $a(ya)$ must have order 1, 2, or 4. If it has order at most 2 for each $ya$ in $K$, then $(2a) \leq L$. But all such d.g. near rings are given in Table 1. Hence, in the present case, we must assume $a(ya)$ has order 4. Obviously, this case arises only if $4|n$. Moreover, $|a(ya)|=4$ implies $(2a)(ya)=(n/2)a$ and $(4a)(ya)=0$. Since $2a \notin L$, $L \neq S$ on $T$ and so $L=(4a)$. Since $|(4a)|=n/4$, there are exactly 8 different row endomorphisms.

Let $c=n/4$. The elements of order 4 in $(a)$ are $\pm ca$. So $a(ya)=\pm ca$. Since the row kernel of $2a$ is $S$ or $T$, $y$ must be odd. Then $(ya)(ya)=y[a(ya)]=y(\pm ca)$. But, since $y$ has the form $4m\pm 1$, $y(\pm ca)=\pm ca$. Thus, as the product of right distributive elements, either $ca$ or $-ca$ is right distributive. Let $Ca$ designate the one which is right distributive.

Since the column kernel of $ya$ is $(4a)$, the column image is isomorphic to $D_8$. Thus $(xa+b)(ya)$ has the form $wa+b$, $w\neq 0$, and is equal to $y[(xa+b)a]$. Hence, since $y$ is odd, $(xa+b)(ya)=(xa+b)a$. Also, $(xa+b)(2ra)=0$ for all values of $x$. Thus an even multiple of $a$ can be right distributive only if it is a right annihilator since its column kernel contains both $S$ and $T$ and so is $D_{2n}$ itself. In particular, if $8|n$ then $c$ is even and $Ca$ is a right annihilator. Then, $0=a(Ca)=Ca^2$. Since $ya^2=a(ya)=\pm ca$, $a^2$ must be a multiple of $a$. Let $a^2=ua$ so that $0=Ca^2=Cua$. Then $n|Cu$ and since $C=\pm n/4$, this implies that $u/4$ is an integer and $a^2\in L$. Since the row image of a contains an element of order 4, the row kernel cannot be $S$ or $T$ but must be a subgroup of $(a)$. Hence $ab\neq 0$, say $ab=va+b$. Then $a(ab)=va^2+va+b\neq 0$ whereas $(aa)b=0$ since $a^2\in L$. This contradiction shows that $c$ cannot be even, that $2^2\| n$.

Since $a(ya)=\pm ca$ which has order 4, the (additive) order of $a^2$ must be multiple of 4. Let $|a^2|=4m$, $m$ odd.

Since $|a(ya)|=4$, we have $(n/2)a=2[a(ya)]=(2a)(ya)$. As noted before, the row image in any $2ra$ row has order at most 2. Hence $(2a)a$ must also be $(n/2)a$. If $z$ is odd, then $(2a)(za)=(2a)a=(n/2)a$. Furthermore, if $za$ is right distributive, then $|a(za)|=4$ and $m|z$. If $2ra$ is right distributive then, as remarked earlier, it is a right annihilator so that $a(2ra)=0$ and $4m|2r$. All told, an element $va$ can be right distributive only if $m|v$.

By the previous paragraph, the $2a$ row image is $\{0, (n/2)a\}$. From earlier, the column kernel of $ta+b$ contains $(2a)$. If $x$ has the same parity as $t$, $(2a)(xa+b)=0$. But, if $x$ has parity different from $t$, $(2a)(xa+b)=(n/2)a$ and $xa+b$ cannot be right distributive since $a(xa+b)$ has order 2. If $xa+b$ is to be right distributive, then it has the form $(2r+t)a+b$. But since $(sa+b)(2ra)=0$ for each $s$ and $r$, $(sa+b)[(2r+t)a+b]=(sa+b)(ta+b)$. Since $\{b, a+b\}$ is a generating set for $D_{2n}$, this shows that the columns of $ta+b$ and $(2r+t)a+b$ are identical. Then, $a[(2r+t)a+b]=2ra^2+a(ta+b)$ implies that $2ra^2=0$ and $4m|2r$ or $m|r$ if $(2r+t)a+b$ is right distributive.

Thus we have shown that all right distributive elements of $N$ are in the subgroup generated by the elements $ma$ and $ta+b$. However, this subgroup is $D_{2n}$ only if $(m, n)=1$. Since $m$ is a factor of $n$, we have a d.g. near ring only if $m=1$. Thus $a^2$ has order $4m=4$, i.e. $a^2=\pm ca$.

Since $a^2$ had order 4 it follows that the product of any two multiples of $a$ is a multiple of $a$ whose order is a divisor of 4. Thus the elements of $(4a)$ are also right annihilators. Recall that $ya$, $y$ odd, is one of the right distributive elements. If $y$ is one more than a multiple of 4 then all elements of the form $(4k+1)a$ are also right distributive; if $y$ is one less than a multiple of 4 then all elements of the form $(4k+3)a$ are also right distributive. In particular, in the first case $a$ is right distributive and in the second case $-a$ is right distributive. Switching notation if necessary, we may presume that the generator of the cyclic group of order $n$ is right distributive. That is, without loss of generality, we presume $a$ is right distributive and $y$ is of the form $4k+1$.

Because $L$ is a (two-sided) ideal, we may consider the d.g. near ring $N/L$. Since $(2a)a=(n/2)a$ in $N$ and $2\|(n/2)$, $\overline{(2a)}\bar{a}\neq\bar{0}$ in $N/L$. Thus the left annihilator of $N/L$ is trivial. Recalling that $(N/L, +)\simeq D_8$ and using Lemma 2, we conclude that $N/L\simeq(D_8; 1)$, the unique d.g. near ring with identity on $D_8$. To avoid confusion with $N$, call the generators of $(D_8; 1)$ by $\alpha$ and $\beta$ instead of $a$ and $b$ and let $K=\{\alpha, \beta\}$. From [7] the multiplication table for $K$ in $(D_8; 1)$ is:

| | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | $\alpha$ | $\beta$ |
| $\beta$ | $\beta$ | $\beta$ |

Note that $\alpha$ is the identity in $(D_8; 1)$ and $3\alpha$ is not right distributive. Thus, under the natural homomorphism from $N$ to $N/L$, the right distributive element $a$ maps to $\bar{a}$ which may be identified with $\alpha$. Since $\bar{a}\cdot\bar{a}=\bar{a}$, it follows that $a^2-a\in(4a)$. Thus $a^2$ is the one of $\pm ca$ whose coefficient is of the form $4k+1$. Hence we set $a^2=Da$ where $D$ is the one of $n/4$ and $3n/4$ which is of the form $4k+1$.

In $N$, without loss of generality, we may take $b$ to be the right distributive element of the form $ta+b$.

Since $(4a)$ is a two-sided annihilator and since, in $(D_8; 1)$, $\beta\alpha=\beta$ and $\alpha\beta=\beta$ and $\beta\beta=\beta$, it follows that $ba=4sa+b$, $ab=4ua+b$, and $bb=4va+b$ for some integers $s$, $u$, and $v$. Consider the $b$ column. In it $(a+b)b=4(u-v)a$. But no element in the $b$ column can have order greater than 2. If $(n/2)a=4(u-v)a$, we obtain the contradiction that $8|n$. Thus $4(u-v)a=0$ and $u=v$. In similar manner we see that $s=v$.

Since $L=(4a)$ is a two-sided annihilator, the column (row) of any element of the form $4wa+b$ is the same as the $b$ column (row). In particular, each element of the form $4wa+b$ is right distributive.

For $n=4$, the multiplication on $N$, is given by the $K$ table for $(D_8; 1)$. For $n$ such that $4\|n$ and $n>4$, the $K$ table is:

| 20 | $a$ | $Da$ | $b$ |
|---|---|---|---|
| $a$ | $Da$ | $Da$ | $b$ |
| $Da$ | $Da$ | $Da$ | $b$ |
| $b$ | $b$ | $b$ | $b$ |

It is easily checked that this table extends to give a left distributive multiplication on $D_{2n}$. Since multiplication is associative on $K$, Lemma 1 guarantees that the multiplication on $D_{2n}$ is associative. Thus $N$ is a d.g. near ring.

In summary, then when $K$ must contain an element of order greater than 2, it follows that $2^2\|n$ and $L=(4a)$. In this case, besides the 19 d.g. near rings which can be defined on any $D_{2n}$, $n$ even, there is exactly one additional d.g. near ring definable. This near ring is such that $N/L\simeq(D_8; 1)$. Only if $n=4$ is $L$ trivial. In that case $N\simeq(D_8; 1)$.

**Table 1**

$L=D_{2n}$

| 1 | $b$ | $a+b$ |
|---|---|---|
| $b$ | 0 | 0 |
| $a+b$ | 0 | 0 |

$L=S$

| 2 | $b$ | $a+b$ |
|---|---|---|
| $b$ | 0 | 0 |
| $a+b$ | 0 | $b$ |

$L=(a)$

| 3 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $b$ | $b$ |
| $a+b$ | $b$ | $b$ |

$L=T$

| 4 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $b$ | 0 |
| $a+b$ | 0 | 0 |

$L=S$

| 5 | $b$ | $a+b$ |
|---|---|---|
| $b$ | 0 | 0 |
| $a+b$ | 0 | $\underline{a}$ |

$L=(a)$

| 6 | $b$ | $a+b$ |
|---|---|---|
| $b$ | 0 | $\underline{a}$ |
| $a+b$ | $a$ | $\underline{a}$ |

$L=S$

| 7 | $b$ | $a+b$ |
|---|---|---|
| $b$ | 0 | 0 |
| $a+b$ | $\underline{a}$ | $\underline{a}$ |

$L=(2a)$

| 8 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $\underline{a}$ | $\underline{a}$ |
| $a+b$ | $\underline{a}$ | 0 |

$L=T$

| 9 | $b$ | $a+b$ |
|---|---|---|
| $b$ | 0 | $\underline{a}$ |
| $a+b$ | 0 | 0 |

$L=(2a)$

| 10 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $\underline{a}$ | 0 |
| $a+b$ | $\underline{a}$ | $\underline{a}$ |

$L=(a)$

| 11 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $\underline{a}$ | $\underline{a}$ |
| $a+b$ | $\underline{a}$ | $\underline{a}$ |

$L=(2a)$

| 12 | $b$ | $a+b$ |
|---|---|---|
| $b$ | 0 | $\underline{a}$ |
| $a+b$ | $\underline{a}$ | 0 |

$L=(2a)$

| 13 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $\underline{a}$ | $0$ |
| $a+b$ | $0$ | $\underline{a}$ |

$L=(2a)$

| 14 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $b$ | $b$ |
| $a+b$ | $b$ | $\underline{a}+b$ |

$L=(2a)$

| 15 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $0$ | $\underline{a}$ |
| $a+b$ | $\underline{a}$ | $b$ |

$L=(2a)$

| 16 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $\underline{a}+b$ | $0$ |
| $a+b$ | $0$ | $\underline{a}$ |

$L=(a)$

| 17 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $b$ | $a+b$ |
| $a+b$ | $b$ | $a+b$ |

$L=S$

| 18 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $0$ | $0$ |
| $a+b$ | $b$ | $a+b$ |

$L=(2a)$

| 19 | $b$ | $a+b$ |
|---|---|---|
| $b$ | $b$ | $0$ |
| $a+b$ | $0$ | $a+b$ |

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## REFERENCES

[1] Ligh, S., *D.G. near rings on certain groups,* Monatsh. Math. 75, 244—249(1971).

[2] Malone, J.J., *D.G. near rings on the infinite dihedral group,* Proc. Roy. Soc. Edinburgh Sect. A 78, 67—60(1977).

[3] __________, and C.G. Lyons, *Finite dihedral groups and d.g. near rings* I, Compositio Math. 24, 305—312(1972).

[4] __________, and __________, *Finite dihedral groups and d.g. near rings* II, Compositio Math. 26, 249—259(1973).

[5] Meldrum, J.D.P., *The endomorphism near-ring of the infinite dihedral group,* Proc. Roy. Soc. Edinburgh Sect. A 76, 311—321(1977).

[6] Pilz, G., *Near-rings,* North-Holland, Amsterdam. 1977.

[7] Willhite, M.L., *Distributively generated near rings on the dihedral group of order eight,* M.S. Thesis, Texas A&M Univ., College Station, Texas, 1970.