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REMARKS ON SOME LOCALIZED SEPARATION AXIOMS AND THEIR IMPLICATIONS

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In 1978, Dube, K.K.; Misra, D.N. [3] have introduced the notation of some localized separation axioms and discussed some of their relations with paracompact spaces, R_D-spaces, almost compact spaces, regularity and normality.

In this paper, we introduce the concept of R_D^* and show that R_D^* is T_0 and R_D but the converse may not be true in general. However, in the case of principal spaces they are equivalent. It has proved that a door space is R_D^*. A maximal and minimal R_D^*-spaces are obtained. Also we generalize some theorems in [3]. Finally we use the concept of localization to define T_i'-distinct ($i=0, 1, 2$) and study the relations between T_i-distinct [3] and T_i'-distinct.

1. Definitions

DEFINITION 1.1. [3]. Let (X, τ) be a topological space.

(i) A point $x \in X$ is *T_0-distinct* if, for any $y \in X$ with $y \neq x$, there exists $U \in \tau$ such that either $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$.

(ii) A point $x \in X$ is *T_1-distinct* if, for any $y \in X$ with $y \neq x$, there exist U, $V \in \tau$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

(iii) A point $x \in X$ is *T_2-distinct* if, for any $y \in X$ with $y \neq x$, there exist U, $V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

DEFINITION 1.2. [4]. A topological space (X, τ) is an *R_D-space* if, for each $x \in X$, $\overline{\{x\}} \cap \{y : x \in \overline{\{y\}}\} = \{x\}$, then $\{x\}'$ is closed.

DEFINITION 1.3. [5]. A topological space (X, τ) is a *door-space* if each sbuset of X is either open or closed.

DEFINITION 1.4. [9]. An open set is minimal at a point $x \in X$ if it contains x and is contained entirely in any open set containing x. If there exists a minimal open set at each point of X, then the topological space is said to be *principal*.

DEFINITION 1.5. [8]. A topological space (X, τ) is a *maximal* (or *minimal*) *Q-space* if it is Q and there is no any topology τ^* on X fine (or weaker) than

τ such that (X, τ^*) is a Q-space.

DEFINITION 1.6. A topological space (X, τ) is an *ultraspace* if the only topology on X finer than τ is the discrete topology. For any two distinct points $y, z \in X$ a topological space (X, τ_{yz}) is a principal ultraspace where τ_{yz} has the basis $\beta_{yz} = \{\{x\}, \{y, z\} : x \in X \text{ and } x \neq z\}$.

DEFINITION 1.7. [7]. Let (X, τ) be a topological space and $A \subset X$.

(i) The set A is *α-almost paracompact* if, every X-open cover of A has an X-open X-locally finite family which refine it and the closures of whose members cover A. The set A is almost paracompact if A is almost paracompact as a subspace.

(ii) the set A is α-nearly paracompact if, every X-open cover of A has an X-open X-locally finite family which refines it and the interiors of the closures of whose members cover A. A is nearly paracompact if A is nearly paracompact as a subspace.

DEFINITION 1.8. [2]. A subset A of a space X is said to be *α-nearly compact* if for every cover $\{U_i\}$ of A by open sets of X, there exist a finite subfamily $\{U_1, U_2, \cdots, U_n\}$ such that $A \subset \bigcup \{\bar{U}_i^{\circ} : i = 1, 2, \cdots, n\}$.

DEFINITION 1.9. [1]. Let (X, τ) be a topological space and $A \subset X$.

(i) $x \in$ X is a *weak limit point* of A if for each open neighbourhood N_x of x, $(\overline{N}_x - \{x\}) \cap A \neq \phi$. The set of all weak limit points of A, denoted by A'', is called the weak derived set of A.

(ii) A is weakly closed if $A'' \subset A$. Weak closedness implies closedness.

2. R_D^*-spaces

DEFINITION 2.1. A topological space (X, τ) is an *R_D^*-space* if, for each $x \in X$, $\overline{\{x\}} \cap \{y : x \in \overline{\{y\}}\} = \{x\}$ and $\{x\}'$ is closed.

It is clear that R_D^* is R_D, but the converse is not true since if $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, then X is R_D but not R_D^*.

THEOREM 2.1. $R_D^* \Longrightarrow T_0$.

PROOF. Let (X, τ) be R_D^* and $x, y \in X$ be two distinct points. Then $\overline{\{x\}} \cap \{z : x \in \overline{\{z\}}\} = \{x\}$, for all $x \in X$. This implies that $y \in \overline{\{x\}}$, $y \notin \{z : x \in \overline{\{z\}}\}$ or $y \notin \overline{\{x\}}$, $y \in \{z : x \in \overline{\{z\}}\}$ or $y \notin \overline{\{x\}}$, $y \notin \{z : x \in \overline{\{z\}}\}$. In these cases, there exists $U \in \tau$ such that $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$. Hence, (X, τ) is T_0.

Now, we give an example to show that the converse of Theorem 2.1 may not be true in general.

EXAMPLE 2.1. Let $X=[-1, 1]$ be a topological space with overlapping interval topology. Then X is T_0, i.e., $\tau=\{[-1, b), (a, b), (a, 1]\}$ where $a<0$, $b>0$. Since $\{b\}$ is T_0-distinct point-set and $\{b\}'=(b, 1]$ which is not closed. Hence, (X, τ) is not R_D^*.

THEOREM 2.2. In the case of a principal topological space T_0 and R_D^* are equivalent.

PROOF. From Theorem (2.1) $R_D^* \Longrightarrow T_0$.

Conversely; from the definition of T_0, we get $\{y : x\in\overline{\{y\}}\}\cap\overline{\{x\}}=\{x\}$. If (X, τ) is a principal T_0 topological space, then $y\in\{x\}'$ implies that $y\in(U_y-U_x)$ where U_x and U_y are the minimal open sets at x and y, respectively, otherwise $U_x=U_y$ and (X, τ) is not T_0. Let $z\in(\{x\}')'$ then there is a point $y\in\{x\}'$ such that $x\in U_y\subset U_z$ where U_z is the minimal open set at z, so $z\in\{x\}'$ and $\{x\}'$ is a closed set. Hence $T_0\Longrightarrow R_D^*$.

REMARK. The derived set of any set of X is closed if X is T_1 as it is shown in the following theorem.

THEOREM 2.3. *Let (X,τ) be a T_1-space, then the derived set A' of each subset A of X is closed.*

PROOF. Let (X,τ) be T_1, $A\subset X$. Let $x\notin A'$, then there exists an open neighbourhood N_x of x such that $N_x\cap A=\{x\}$ or ϕ. If $N_x\cap A=\phi$. So $N_x\cap A'=\phi$ and if $N_x\cap A=\{x\}$, we assume that there exists a point $y\in(N_x\cap A')$. Then $y\neq x$ because $x\notin A'$. Since (X,τ) is T_1, $N_x-\{x\}$ is an open set containing y and hence $(N_x-\{x\})\cap A\neq\phi$. This contradicts to $N_x\cap A=\{x\}$. So $N_x\cap A'=\phi$. Hence, A' is closed.

THEOREM 2.4. *Door*$\Longrightarrow R_D^*$.

PROOF. Let (X,τ) be a door space then for any point $x\in X$, $\{x\}\in\tau$ or $(X-\{x\})\in\tau$. If $(X-\{x\})\in\tau$, $\overline{\{x\}}=\{x\}$ and $\{x\}'=\phi$ and if $\{x\}\in\tau$, $\{y : x\in\overline{\{y\}}\}=\{x\}$ and $\{x\}'=(X-\{x\})\cap\overline{\{x\}}$. Then in both cases $\{y : x\in\overline{\{y\}}\}\cap\overline{\{x\}}=\{x\}$ and $\{x\}'$ is a closed set. Hence (X,τ) is an R_D^*-space.

THEOREM 2.5. *Each principal ultraspace (X,τ_{yz}), where $y, z\in X$ are any two distinct points of X is a maximal R_D^*-space.*

PROOF. The proof is obvious. We construct examples of a minimal R_D^*-space. Let X be a non-empty set, $p, q \in X$ be two distinct points and $\hat{\tau}$ be a topology on X such that:

1- $\{p\}$, $(X-\{q\}) \in \hat{\tau}$ and

2- $G \in \hat{\tau}$, $G \neq \phi$ and $G \neq X$ implies that there are two open sets, $V, W \in \hat{\tau}$ such that $V \subset G \subset W$ and $G = V \cup \{x\}$, $x \in (X-(V \cup \{q\}))$ and $W = G \cup \{y\}$, $y \in (X-(G \cup \{q\}))$ with the following conditions:

(i) there is no open set $G^* \in \hat{\tau}$, $G^* \neq G$ such that $V \subset G^*$ or $G^* \subset W$,

(ii) there is no open set $W^* \in \hat{\tau}$, $W^* \neq W$ such that $G \subset W^*$ and

(iii) there is no open set $V^* \in \hat{\tau}$, $V^* \neq V$ such that $V^* \subset G$.

If X is a countable set then $\hat{\tau}$ may take the form: $\hat{\tau} = \{X, \phi, A_i : i \geq 1\}$, where $A_1 = \{p\}$ and $A_{i+1} = A_i \cup \{x\}$, $x \in (X-(A_i \cup \{q\}))$.

This construction topology is a principal topology and it has the properties:
1- each open set is minimal at a point of X,
2- for each $x \in X$ there is an open set $G \in \hat{\tau}$ such that G is minimal at x and
3- let $x, y \in X$ be two distinct points and G_x, $G_y \in \hat{\tau}$ be the minimal open sets at x and y, respectively, then $G_x \neq G_y$.

THEOREM 2.6. *A space $(X, \hat{\tau})$ where $\hat{\tau}$ is the above constructed topology is a minimal R_D^*-space.*

PROOF. It is not difficult to show that $(X, \hat{\tau})$ is T_0 and so it is an R_D^*-space. Let τ^* be a topology on X which is weaker than $\hat{\tau}$ then there is an open set $G \in \hat{\tau}$ such that $G \notin \tau^*$. Since $G \in \hat{\tau}$, there is an open set $W \in \hat{\tau}$ such that $G \subset W$ and $W \in \tau^*$. If $G \in \hat{\tau}$ is minimal at $x_0 \in X$ and $W \in \hat{\tau}$ is minimal at $y_0 \in X$ hence $W \in \hat{\tau}$ is a minimal open set at both x_0 and y_0, so each open set in τ^* containing x_0 contains y_0. Therefore (X, τ^*) is not T_0 and hence it is not R_D^*. Then $(X, \hat{\tau})$ is a minimal R_D^*-space.

REMARK. It is clear that $(X, \hat{\tau})$ is a minimal T_0-space.

3. Generalizations of some theorems in [3]

In [3, Theorem 3.2], Dube and Misra showed that if, (X, τ) is paracompact (with no separation axioms assumed) and if x is a T_2-distinct point disjoint from a closed set F, then x and F have disjoint open neighbourhoods. This result can be considered as a special case of the following:

THEOREM 3.1. *Let $A \subset X$ be any α-nearly paracompact subset and $x \in X$ be a*

T_2-distinct point disjoint with A, then there exist two disjoint open neighbourhoods for x and A.

PROOF. Let $A \subset X$ be any α-nearly paracompact subset and $x \in X-A$ is T_2-distinct point. Then $x \notin y$, for every $y \in A$. This implies that there exists an open neighbourhood U_y of y such that $x \notin \overline{U}_y$. The family $\{U_y : y \in A\}$ is an open cover of A, then there exists a X-locally finite of X-open sets $V=\{V_\beta : \beta \in j\}$ which refines $\{U_y : y \in A\}$ and such that $A \subset \bigcup \{\overline{v}_\beta^\circ : \beta \in j\}$. Let $G=\bigcup \{\overline{v}_\beta^\circ : \beta \in j\}$. Then G is an open set containing A and $x \notin G$. Therefore, there exists an open neighbourhood N_x of x intersects with a finite number members V_{β_1}, V_{β_2}, $\cdots$, V_{β_n} of members V. So, there exist the corresponding open neighbourhoods W_1, W_2, $\cdots$, W_n of x such that $W_i \cap V_{\beta_i}=\phi$, for all $i=1, 2, \cdots$, n. Let $N_x \cap (\bigcap_{i=1}^{n} W_i)=V_x$, so $V_x \cap V_\beta=\phi$ for all $V_\beta \in V$, then $V_x \cap \overline{V}_\beta^\circ=\phi$. This implies that $V_x \cap G=\phi$. Hence, V_x and G are two open neighbourhoods of x and A, respectively.

Dube and Misra in [3], have proved that if, in a topological space (X, τ), an almost compact set and a T_2-distinct point-set form a partition of X, then the almost compact set is closed. This result can be strengthed as follows:

THEOREM 3.2. *If, in X, an α-almost paracompact set and a T_2-distinct point-set form a partition of X, then the α-almost paracompact set is closed.*

PROOF. Let $A=\{y_i : i \in I\}$ be an α-almost paracompact subset of X and A^c is T_2-distinct point-set. Take any point $x \in A^c$, then $x \neq y_i$, for each $i \in I$. So, there exist two families of open neighbourhoods: $U=\{U_i : x \in U_i,\ i \in I\}$ and $V=\{V_i : y \in V_i,\ i \in I\}$ of x and y_i, respectively, such that $U_i \cap V_i=\phi$, for each $i \in I$. One may consider V as an open cover of A, and since A is α-almost paracompact, then there exists a locally finite family $H=\{V_\beta : \beta \in j\}$ of open sets refining V and $A \subset \bigcup_{\beta \in j} \overline{V}_\beta$. Also there exist an open neighbourhood N_x of x intersects a finite members of H, then $N_x \cap (\bigcap_{i \in I_0} U_i)=V_x$, where I_0 is a finite subset of I. This implies that $V_x \cap (\bigcup_{\beta \in j} \overline{V}_\beta)=\phi$, so $V_x \cap A=\phi$. Hence A is closed.

COROLLARY. *If A is α-nearly paracompact, in the above theorem, then A is weakly closed.*

4. New localized separation axioms

Mashhour [6], introduced and studied the separation axioms T_i', $(i=0, 1, 2)$ and T_1''. The definitions of these axioms are based on those of the basic lower separation axioms. We localize some of these axioms as follows.

DEFINITION 4.1. Let (X, τ) be a topological space.

(i) A point $x \in X$ is *T_0'-distinct* if, for any $y \in X$ with $y \neq x$ there exists an open neighbourhood of one of them to which the other is a boundary point.

(ii) A point $x \in X$ is *T_1'-distinct* if, for any $y \in X$ with $y \neq x$, there is an open neighbourhood for one point, say x, to which y is a boundary point, and an open neighbourhood of y to which x does not belong.

(iii) A point $x \in X$ is *T_2'-distinct* if, for any $y \in X$ with $y \neq x$, there exist two open neighbourhoods U and V of x and y, respectively, such that $\overline{U} \cap \overline{V} = \phi$.

THEOREM 4.1. *If A is an α-nearly compact subset of X and $x \in X$ is T_2'-distinct point disjoint with A, then there exist disjoint closed neighbourhoods of x and A.*

PROOF. It is similar to that of Theorem 3.1.

COROLLARY. *Every pair of disjoint α-nearly compact subsets, one of which consists of T_2'-distinct points has disjoint closed neighbourhoods.*

THEOREM 4.2. *Let (X, τ) be a topological space in which every nonempty open set is dense and $x \in X$ be T_0-distinct point, then x is T_0'-distinct.*

PROOF. Let $x \in X$ be T_0-distinct point, then for every point $y \in X$, $y \neq x$ there exists an open neighbourhood U of one point of them which does not contain the other. Since $\overline{U} = X$, then the other is a boundary point of U. Hence, x is T_0'-distinct.

THEOREM 4.3. *Let (X, τ) be a topological space, $x \in X$ be T_0'-distinct point which is also T_1-distinct, then x is T_1'-distinct.*

PROOF. Let x be T_0'-distinct point which is also T_1-distinct, then for every $y \in X$, $y \neq x$ there exists an open neighbourhood U of one of them, say x, to which y is a boundary. Since x is T_1-distinct and $y \neq x$, then there exists an open neighbourhood V of y which does not contain x. Hence, x is T_1'-distinct.

THEOREM 4.4. *Let x be T_1-distinct point in a connected topological space (X, τ) then x is T_1'-distinct.*

PROOF. Let x be T_1-distinct point in X. For every $y \in X$, $y \neq x$, $X - \{x\}$ is an open neighbourhood of y. Since (X, τ) is connected, then $\overline{X - \{x\}} = X$. So, x is a boundary point of $X - \{x\}$. Also there exists an open neighbourhood of x which does not contain y. Hence, x is T_1'-distinct.

THEOREM 4.5. *Let x be T_0-distinct point in a regular space X, then x is T_2'-distinct.*

PROOF. Let X be a regular space, x is T_0-distinct point in X. Then, there is an open neighbourhood U of one point, say x, such that $y \notin U$. Since X is regular, then there is a weakly closed neighbourhood V of x such that $V \subset U$. Then, $X - V$ is open neighbourhood of y and $x \notin X - V$. Since X is regular, x is T_2'-distinct.

THEOREM 4.6. *Let x be T_1-distinct point in a normal space X, then x is T_2'-distinct.*

PROOF. It is similar to that of the above theorem.

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