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A NOTE ON PAIRWISE s-NORMAL SPACES

By Takashi Noiri

1. Introduction

In 1963, N. Levine [4] introduced the concept of semi-open sets in a topological space. Recently, by using semi-open sets S.N. Maheshwari and R. Prasad [5] have defined the concept of pairwise s-normality which is weaker than that of pairwise normality due to J.C. Kelly [3]. On the other hand, R. Anantharaman and M.G. Murdeshwar [2] introduced a new class of functions called optimally continuous and showed that such functions are implied by homeomorphisms and imply continuous, open and closed surjections. The purpose of the present note is to show that pairwise s-normality is preserved under pairwise optimally continuous surjections, however, it is not preserved under pairwise continuous, pairwise open and pairwise closed surjections.

2. Preliminaries

Throughout the present note (X,σ) and (Y,τ) denote topological spaces, (X, σ_1, σ_2) and (Y, τ_1, τ_2) mean bitopological spaces. A subset S of (X,σ) is said to be *semi-open* [4] if there exists an open set $U\in\sigma$ such that $U\subset S\subset \mathrm{Cl}(U)$, where $\mathrm{Cl}(U)$ denotes the closure of U. The family of all semi-open sets in (X, σ) will be denoted by $SO(X, \sigma)$.

DEFINITION 2.1. A surjection $f:(X, \sigma)\rightarrow(Y, \tau)$ is said to be *optimally continuous* [2] if σ is the pre-image topology.

REMARK 2.2. Every homeomorphism is optimally continuous, but not conversely [2, p.503].

The following two lemmas, due to R. Anantharaman and M.G. Murdeshwar [2], are very useful in the sequel.

LEMMA 2.3. *Every optimally continuous surjection is continuous, open and closed, but not conversely.*

LEMMA 2.4. *If a surjection $f:(X, \sigma)\rightarrow(Y, \tau)$ is optimally continuous and F is closed in (X, σ), then $f^{-1}(f(F))=F$.*

LEMMA 2.5. (Noiri, [7]). *If a surjection* $f:(X, \sigma)\to(Y, \tau)$ *is optimally continuous, then for any subset* $S\subset Y$ *and any* $U\subset SO(X, \sigma)$ *containing* $f^{-1}(S)$, *there exists* $V\in SO(Y, \tau)$ *such that* $S\subset V$ *and* $f^{-1}(V)\subset U$.

3. Pairwise s-normal spaces

DEFINITION 3.1. A bitopological space (X, σ_1, σ_2) is said to be *pairwise s-normal* [5] if for every σ_i-closed set A and σ_j-closed set B such that $A\cap B=\phi$, there exist $U\in SO(X, \sigma_j)$ and $V\in SO(X, \sigma_i)$ such that $A\subset U$, $B\subset V$ and $U\cap V=\phi$, where $i, j=1, 2, i\neq j$.

DEFINITION 3.2. A function $f:(X, \sigma_1, \sigma_2)\to(Y, \tau_1, \tau_2)$ is said to be *pairwise continuous* (resp. *pairwise open, pairwise closed*) if the induced functions $f:(X, \sigma_1)\to(Y, \tau_1)$ and $f:(X, \sigma_2)\to(Y, \tau_2)$ are continuous (resp. open, closed).

DEFINITION 3.3. A surjection $f:(X, \sigma_1, \sigma_2)\to(Y, \tau_1, \tau_2)$ is said to be *pairwise optimally continuous* if the surjections $f:(x, \sigma_1)\to(Y, \tau_1)$ and $f:(X, \sigma_2)\to(Y, \tau_2)$ are optimally continuous.

REMARK 3.4. Pairwise *s*-normality is not preserved under pairwise continuous, pairwise open and pairwise closed surjections as shown by the following example.

EXAMPLE 3.5. Let $X=\{a, b, c, d\}$ and define topologies for X as follows:

$$\sigma_1=\{X, \{a, b, d\}, \{b, c, d\}, \{b, d\}, \{b\}, \{d\}, \phi\},$$
$$\sigma_2=\{X, \{b, c, d\}, \{b, d\}, \{b\}, \{d\}, \phi\}.$$

Let $Y=\{a, b, c\}$ and define topologies for Y as follows:

$$\tau_1=\{Y, \{a, b\}, \{a, c\}, \{a\}, \phi\} \text{ and } \tau_2=\{Y, \{a, c\}, \{a\}, \phi\}.$$

Define a function $f:(X, \sigma_1, \sigma_2)\to(Y, \tau_1, \tau_2)$ as follows: $f(a)=b$, $f(b)=f(d)=a$ and $f(c)=c$. Then (X, σ_1, σ_2) is pairwise s-normal, but it is not pairwise normal [5, Example 1]. It is obvious that (Y, τ_1, τ_2) is not pairwise *s*-normal. Although f is a pairwise continuous, pairwise open and pairwise closed surjection, it is not pairwise optimally continuous.

THEOREM 3.6. *Let a surjection* $f:(X, \sigma_1, \sigma_2)\to(Y, \tau_1, \tau_2)$ *be pairwise optimally continuous. Then,* (X, σ_1, σ_2) *is pairwise s-normal if and only if* (Y, τ_1, τ_2) *is pairwise s-normal.*

PROOF. *Necessity*. Suppose that (X, σ_1, σ_2) is pairwise *s*-normal. Let B_i and B_j be disjoint sets of Y such that B_i is τ_i-closed and B_j is τ_j-closed, $i\neq j$, i,

$j=1, 2$. By Lemma 2.3, f is pairwise continuous and hence $f^{-1}(B_i)$ and $f^{-1}(B_j)$ are disjoint sets of X such that $f^{-1}(B_i)$ is σ_i-closed and $f^{-1}(B_j)$ is σ_j-closed. Since (X, σ_1, σ_2) is pairwise s-normal, there exist disjoint semi-open sets $U_j \in SO(X, \sigma_j)$ and $U_i \in SO(X, \sigma_i)$ such that $f^{-1}(B_i) \subset U_j$ and $f^{-1}(B_j) \subset U_i$. Moreover, by Lemma 2.5 there exist $V_j \in SO(Y, \tau_j)$ and $V_i \in SO(Y, \tau_i)$ such that $B_i \subset V_j$, $B_j \subset V_i$, $f^{-1}(V_j) \subset U_j$ and $f^{-1}(V_i) \subset U_i$. Since f is surjective and $U_i \cap U_j = \phi$, we have $V_i \cap V_j = \phi$. This shows that (Y, τ_1, τ_2) is pairwise s-normal.

Sufficiency. Suppose that (Y, τ_1, τ_2) is pairwise s-normal. Let A_i and A_j be disjoint sets of X such that A_i is σ_i-closed and A_j is σ_j-closed, $i \neq j$, $i, j = 1, 2$. By Lemmas 2.3 and 2.4, $f(A_i)$ and $f(A_j)$ are disjoint sets of X such that $f(A_i)$ is τ_i-closed and $f(A_j)$ is τ_j-closed. Since (Y, τ_1, τ_2) is pairwise s-normal, there exist disjoint semi-open sets $V_j \in SO(Y, \tau_j)$ and $V_i \in SO(Y, \tau_i)$ such that $f(A_i) \subset V_j$ and $f(A_j) \subset V_i$. By Lemma 2.3, f is pairwise continuous pairwise open and hence $f^{-1}(V_j) \in SO(X, \sigma_j)$ and $f^{-1}(V_i) \in SO(X, \sigma_i)$ [1, Theorem 1]. Moreover, we have $A_i \subset f^{-1}(V_j)$, $A_j \subset f^{-1}(V_i)$ and $f^{-1}(V_j) \cap f^{-1}(V_i) = \phi$. This shows that (X, σ_1, σ_2) is pairwise s-normal.

The complement of a semi-open set is said to be *semi-closed*. A function $f:(X, \sigma) \rightarrow (Y, \tau)$ is said to be *semi-closed* [6] if the image of every closed set of (X, σ) is semi-closed in (Y, τ).

DEFINITION 3.7. A function $f:(X, \sigma_1, \sigma_2) \rightarrow (Y, \tau_1, \tau_2)$ is said to be *pairwise semi-closed* if the functions $f:(X, \sigma_1) \rightarrow (Y, \tau_1)$ and $f:(X, \sigma_2) \rightarrow (Y, \tau_2)$ are semi-closed.

THEOREM 3.8. *Let $f:(X, \sigma_1, \sigma_2) \rightarrow (Y, \tau_1, \tau_2)$ be a pairwise continuous and pairwise semi-closed surjection. If (X, σ_1, σ_2) is pairwise normal, then (Y, τ_1, τ_2) is pairwise s-normal.*

PROOF. Make use of [6, Theorem 5] instead of Lemma 2.5, and the proof is similar to that of Necessity of Theorem 3.6.

REMARK 3.9. Quite recently V. Popa [8] has defined a function $f:(X, \sigma_1, \sigma_2) \rightarrow (Y, \tau_1, \tau_2)$ to be *pairwise semi-closed* if for each $i=1, 2$ the image of each semi-closed set in (X, σ_i) is a semi-closed set of (Y, τ_i) and showed that pairwise s-normality is preserved under pairwise continuous pairwise semi-closed surjections.

Yatsushiro College of Technology
Yatsushiro, Kumamoto
866 Japan

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