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SOME GENERATING FUNCTIONS FOR A NEW CLASS OF POLYNOMIALS

By A.N. Srivastava

This is the continuation of the study of the polynomials $A_{rn}^{\alpha,a}(x,p,s)$ introduced by the present author and Singh. These polynomials are defined by means of following n^{th} differential formula

$$A_{rn}^{\alpha,a}(x,p,s)=x^{-\alpha}\exp(px^r)\ {}_aT_s^n(x^\alpha\exp(-px^r)),$$

where

$${}_aT_s\equiv x^a(s+xD),\ D\equiv d/dx.$$

In the present paper we obtain a bilinear and a class of bilateral generating function for these polynomials. The bilateral generating function is then extended in the form of mixed trilateral generating function. Some particular cases have also been noticed.

1. Introduction

In a recent paper [5] we have introduced a new class of polynomials by means of generalized Rodrigue's formula

$$A_{rn}^{\alpha,a}(x,p,s)=x^{-\alpha}\exp(px^r)\ {}_aT_s^n(x^\alpha\exp(-px^r)), \tag{1.1}$$

where

(1.2) ${}_aT_s\equiv x^a(s+xD)$, $D\equiv d/dx$, a and s being constants. The generalized polynomial set (1.1) provides as a special case several classical polynomials and their recent generalizations.
For example

$$H_n(x)=(-1)^n A_n^{0,-1}(x,1,0), \tag{1.3}$$

where $H_n(x)$ are Hermite polynomials [3].

$$H_n^r(x,\alpha,p)=(-1)^n A_{rn}^{\alpha,-1}(x,p,0), \tag{1.4}$$

where $H_n^r(x,\alpha,p)$ are generalized Hermite polynomials due to Gould and Hopper [2].

$$L_n^{(\alpha)}(x)=\frac{1}{n!}x^{-n}A_n^{\alpha,-1}(x,1,0) \tag{1.5}$$

where $L_n^{(\alpha)}(x)$ are generalized Laguerre polynomials.

In [5] we obtained operational formulas and generating functions for these polynomials. In the present paper we establish a bilinear and a class of bilateral generating function for these polynomials. The bilateral generating function is then extended in the form of mixed trilateral generating function.

2. Bilinear generating function

We write (1.1) as

$$(2.1)\qquad A_n^{\alpha,a}(x)=x^{-\alpha}\exp(px^r)\ {}_aT_s^n\ (x^\alpha\exp(-px^r)),$$

where and throughout the present paper,

$$(2.2)\qquad \mathrm{A}_n^{\alpha,a}(x)=A_n^{\alpha,a}(x,r,p,s).$$

Consider

$$\sum_{n=0}^{\infty}\frac{(y)_n t^n}{(\alpha+1)_n(\beta+1)_n n\,!}A_n^{\alpha,a}(x)A_n^{\beta,a}(y)$$

$$=\sum_{n=0}^{\infty}\frac{(y)_n t^n}{(\alpha+1)_n(\beta+1)_n n\,!}x^{-\alpha}\exp(px^r)_aT_s^n(x^\alpha\exp(-px^r))$$

$$y^{-\beta}\exp(py^r)_aT_s^n(y^\beta\exp(-py^r)).$$

$$=x^{-\alpha}y^{-\beta}\exp\{p(x^r+y^r)\}\sum_{n=0}^{\infty}\frac{(y)_n t^n}{(\alpha+1)_n(\beta+1)_n n\,!}$$

$$\cdot{}_aT_s^n(x^\alpha\exp(-px^r))\ {}_aT_s^n(y^\beta\exp(-py^r))$$

$$=x^{-\alpha}y^{-\beta}\exp\{p(x^r+y^r)\}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i\,!\,j\,!}p^{i+j}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(y)_n t^n}{(\alpha+1)_n(\beta+1)_n n!}{}_aT_s^n\,(x^{\alpha+ri})_aT_s^n(y^{\beta+rj}).$$

Now making use of the result [5], ${}_aT_s^n(x^\alpha)=a^n\left(\frac{\alpha+s}{a}\right)_n x^{\alpha+an}$ where α is arbitrary, we obtain

$$\sum_{n=0}^{\infty}\frac{(y)_n t^n}{(\alpha+1)_n(\beta+1)_n n\,!}A_n^{\alpha,a}(x)A_n^{\beta,a}(y)$$

$$=\exp\{p(x^r+y^r)\}\sum_{i,j}\frac{(-1)^{i+j}}{i\,!\,j\,!}(x^rp)^i(y^rp)^j.$$

$$F\left[\begin{matrix}\frac{\alpha+ri+s}{a}, & \frac{\beta+rj+s}{a} \\ \alpha+1, & \beta+1\end{matrix} ; a^2 t\right]. \tag{2.3}$$

Putting $p=r=a=1$, $s=0$ in (2.3) we obtain a bilinear generating relation for generalized Laguerre polynomials due to Chatterjea [1].

3. Bilateral generating function

In [5] we have obtained the following generating relation,

$$\sum_{n=0}^{\infty} A^{\alpha,a}_{(n+m)}(x)t^n/n! \tag{3.1}$$
$$=(1-ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \exp[px^r\{1-(1-ax^a t)^{-r/a}\}] \cdot A^{\alpha,a}_m(x(1-ax^a t)^{-1/a}).$$

Now we establish a bilateral generating function for these polynomials in the form of following theorem.

THEOREM 1. *For the polynomials* $A^{\alpha,a}_n(x)$ *generated by* (3.1), *if*

$$F_q[x,t]=\sum_{n=0}^{\infty} K_n A^{\alpha,a}_{qn+m}(x)t^n \tag{3.2}$$

then

$$\sum_{n=0}^{\infty} A_{n+m}(x)\sigma^q_n(y)\,t^n \tag{3.3}$$
$$=(1-ax^a t)^{-(\alpha+s)/a} \exp[px^r\{1-(1-ax^a t)^{-r/a}\}]F_q[x(1-ax^a t)^{-1/a}, yt^q]$$

where

$$\sigma^q_n(y)=\sum_{i=0}^{[n/q]} K_i \binom{m+n}{m+qi} y^i \tag{3.4}$$

and q *is an arbitrary positive integer,* $K_n \neq 0$ *are arbitrary constants.*

PROOF OF THE THEOREM. Substituting for the polynomials $\sigma_n^q(y)$ from (3.4) in the left hand side of (3.3), we obtain

$$\sum_{n=0}^{\infty} A^{\alpha,a}_{n+m}(x)\sigma^q_n(y)t^n$$
$$=\sum_{n=0}^{\infty} A^{\alpha,a}_{n+m}(x)t^n \sum_{i=0}^{[n/q]} K_i \binom{m+n}{m+qi} y^i$$
$$=\sum_{i=0}^{\infty} K_i y^i t^{qi} \sum_{n=0}^{\infty} \binom{m+n+qi}{m+qi} A^{\alpha,a}_{n+qi+m}(x)t^n$$

$$=(1-ax^a t)^{-(\alpha+s)/a}\exp[px^r\{1-(1-ax^a t)^{-r/a}\}]$$

$$\cdot\sum_{i=0}^{\infty}K_i A^{\alpha,a}_{qi+m}(x(1-ax^a t)^{-1/a})(yt^q)^i,$$

by using the generating relation (3.1). Now the theorem would follow at once if we interpret this last expression by means of (3.2).

PARTICULAR CASES.

(i) In view of the relationship (1.4), the relation (3.3) reduces to the form

$$(3.5)\qquad \sum_{n=0}^{\infty}H^r_{n+m}(x,\alpha,p)\sigma_n^{\,q}(y)t^n=(-1)^m(1-t/x)^\alpha\exp[px^r\{1-(1-t/x)^r\}]$$

$$\cdot F_q[x(1-t/x),\quad yt^q],$$

which seems to be a new result for Gould Hoppers' polynomials.

(ii) Putting $q=1$ and $m=0$ in (3.3) and making use of the relation (1.5), we obtain a bilateral generating relation for Laguerre polynomials due to Singhal and Srivastava [4]. Similarly by assigning suitable values to the parameters in $A^{\alpha,a}_n(x)$, we can obtain a number of bilateral generating relations for several class of polynomials.

4. Mixed trilateral generating function

We now establish extension of Theorem 1 in the form of mixed trilateral generating function.

THEOREM 2. *For the polynomials* $A^{\alpha,a}_n(x)$ *generated by* (3.1), *if*

$$(4.1)\qquad F_q[x,y,t]=\sum_{n=0}^{\infty}K_n A^{\alpha,a}_{qn+m}(x)g_n(y)t^n$$

then

$$(4.2)\qquad \sum_{n=0}^{\infty}A_{n+m}(x)\sigma^q_n(y,z)t^n=(1-ax^a t)^{-(\alpha+s)/a}\exp[px^r\{1-(1-ax^a t)^{-r/a}\}]$$

$$\cdot F_q[x(1-ax^a t)^{-1/a},\ y, zt^q]$$

where

$$(4.3)\qquad \sigma^q_n(y,z)=\sum_{i=0}^{[n/q]}K_i\binom{m+n}{m+qi}gi(y)z^i$$

and q *is an arbitrary positive integer,* $K_i\neq 0$ *are arbitrary* constants.

We omit the proof of this theorem as it would run exactly parallel to what we have outlined in the proof of Theorem 1.

As an application of (4.2) by giving suitable values to the arbitrary coefficients K_n, we can obtain a large variety of mixed trilateral generating functions for several polynomials and their recent generalizations.

Department of Mathematics
Banaras Hindu University
Varanasi-221005
INDIA

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