Kyungpook Math. J.
Volume 22, Number 1
June, 1982

FIXED POINTS IN L-SPACES

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0. Introduction

Recently Ćirić [1] proved some fixed point theorems when the mapping f of a metric space (M,d) satisfies the following inequality.

(1) $\quad d(fx,fy)\leq\alpha\max\{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{1}{2}[d(x,fy)+d(y,fx)]\}$

for $x,y\in M$ and for $0\leq\alpha<1$. He termed this type of mapping as generalized contraction.

This paper consists of two sections. In section 1, we have proved a fixed point theorem in spaces of type L of Fréchet, which we shall called separated L-spaces when the mapping f satisfies on the following condition

(2) $\quad d(fx,fy)\leq\alpha\max\{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{1}{2}[d(x,fy)+d(y,fx)]\}$

for $x,y\in M$ and $0\leq\alpha<1$. Similar result in non-separated L-spaces will be also stated. In section 2, we have extended Theorem 1.1 for family of mappings.

1. Mapping in L-spaces

Before going in to the theorems we state the following definitions.

DEFINITION 1. Let ω denote the set of all non-negative integers. A pair $(M,\rightarrow)$ of a set M and a sub-set $\rightarrow$ of the set $M^{\omega}\times M$ is called an *L-space* if the following conditions are satisfied:

(L-1) $\quad$ If $x_n=x\in M$ for all $n\in\omega$, then $(\{x_n\}_{n\in\omega},x)\in\rightarrow$.

(L-2) $\quad$ If $(\{x_n\}_{n\in\omega},x)\in\rightarrow$, then $(\{x_{n_i}\}_{i\in\omega},x)\in\rightarrow$

for every sub-sequence $\{x_{n_i}\}_{i\in\omega}$ of $\{x_n\}_{n\in\omega}$. In what follows, we shall write $\{x_n\}_{n\in\omega}\rightarrow x$ or $x_n\rightarrow x$ instead of $(\{x_n\}_{n\in\omega},\ x)\in\rightarrow$, and read $\{x_n\}_{n\in\omega}$ converges to x.

DEFINITION 2. Let $(M,\rightarrow)$ be an L-space. It is said to be *separated* if each sequence in M converges to at most one point of M.

DEFINITION 3. Let d be a non-negative extended real valued function on $M\times M$: $0\leq d(x,y)\leq\infty$ for all $x,y\in M$. The L-space $(M,\rightarrow)$ is said to be *d-complete* if each sequence $\{x_n\}_{n\in\omega}$ in M with $\sum_{n=0}^{\infty}d(x_{n+1},\ x_n)<\infty$ converges to at most one point of M.

LEMMA ([2]). *Let $(M,\rightarrow)$ be an L-space which is d-complete for a non-negative extended real valued function d on $M\times M$. If $(M,\rightarrow)$ is separated, then $d(x,y)=d(y,x)=0$ implies $x=y$ for every x,y in M.*

THEOREM 1.1. *Let $(M,\rightarrow)$ be a separated L-space which is d-complete for a non-negative extended real valued function d on $M\times M$ and f be a continuous mapping of M into itself satisfying the following conditions for some α,β with $0\leq\alpha<1$ and $0<\beta\leq\infty$.*

(3) $d(fx,fy)\leq\alpha\max\left\{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{1}{2}[d(x,fy)+d(y,fx)]\right\}$ *for every $x,y\in M$ with $d(x,y)<\beta$.*

(4) $d(fb,b)<\beta$ *for some $b\in M$.*

Then f has a fixed point, and the sequence $\{f^n b\}_{n\in\omega}$ converges to the fixed point; if in addition

(5) $d(x,y)<\beta$ *for all fixed points $x,y\in M$ of f, then f has a unique fixed point in M.*

PROOF. Now

$$d(f^{n+1}b,f^nb)\leq\alpha\max\{d(f^nb,f^{n-1}b),\ d(f^nb,f^{n+1}b),\ d(f^nb,f^{n-1}b),\ \frac{1}{2}[d(f^{n-1}b,f^{n+1}b)+d(f^nb,f^nb)]$$
$$\leq\alpha\max\left\{d(f^nb,f^{n-1}b),\ \frac{1}{2}d(f^{n-1}b,f^{n+1}b)\right\}.$$

If the maximum is $d(f^{n-1}b,f^nb)$ we have

$$d(f^{n+1}b,f^nb)\leq\alpha d(f^{n-1}b,f^nb).$$

If the maximum is $\frac{1}{2}d(f^{n+1}b,f^nb)$, then we get

$$d(f^{n+1}b,f^nb)\leq\frac{\frac{1}{2}\alpha}{1-\frac{1}{2}\alpha}d(f^nb,f^{n-1}b).$$

Let $\theta=\max\left\{\alpha,\ \frac{\frac{1}{2}\alpha}{1-\frac{1}{2}\alpha}\right\}$. Then $\theta<1$ and we have

$$d(f^{n+1}b,f^nb)\leq\theta\, d(f^{n-1}b,f^nb).$$

By induction we get

$$d(f^{n+1}b,f^nb)\leq\theta^n d(fb,b),$$

for every $n\in\omega$, and so we have $\sum_{n=0}^{\infty} d(f^{n+1}b, f^n b)<\infty$. Hence the d-completeness of the space implies that the sequence $\{f^n b\}_{n\in\omega}$ converges to some $u\in M$. So, by the continuiy of f, there is a subsequence $\{f^{n_i}b\}_{i\in\omega}$ of $\{f^n b\}_{n\in\omega}$ such that $f(f^{n_i}b)\to f(u)$. But then since $\{f(f^{n_i}b)\}_{i\in\omega}$ is a subsequence of $\{f^n(b)\}_{n\in\omega}$, we have $f(f^{n_i}b)\to u$. Therefore $fu=u$. To prove the unicity it will suffice to show that f has at most one fixed point under the condition (5). If $u, v\in M$ be fixed points of f. Then, since $d(u, v)<\beta$, we have $d(u, v)=d(fu, fv)\leq\alpha d(u, v)$ and so $d(u, v)=0$. Since $d(v, u)<\beta$, this implies $d(v, u)=0$. Therefore $v=u$ by the Lemma.

We now proceed to prove a theorem for non-separated L-spaces.

THEOREM 1.2. *Let $(M, \to)$ be an L-space which is d-complete for a continuous non-negative extended real valued function d on the product space $M\times M$ with the property that $d(x, y)=0$ implies $x=y$. If f is a continuous mapping of M into itself satisfying the conditions* (3) *and* (4) *of Theorem* 1.1 *for some α, β with $0<\alpha<1$ and $0<\beta<\infty$, then f has a fixed point. If in addition condition* (5) *is satisfied, then the fixed point is unique also.*

PROOF. By the induction (as in Theorem 1) we have

$$(6)\quad d(f^{n+1}b, f^n b)\leq\delta^n d(fb, b), \quad \left[\delta=\max\left\{\alpha, \frac{\frac{1}{2}\alpha}{1-\frac{1}{2}\alpha}\right\}<1\right]$$

for every $n\in\omega$. Hence the same argument employed in the proof of Theorem 1.1 yields that the sequence $\{f^n b\}_{n\in\omega}$ converges to some $u\in M$ and that $f(f^{n(k)} b)\to f(u)$ for some subsequence $\{f^{n(k)}b\}_{k\in\omega}$ of the sequence $\{f^n b\}_{n\in\omega}$. Therefore the continuity of f implies that

$d(f(f^{n(k_i)}b), f^{n(k_i)}b)\to d(fu, u)$ for some subsequence $\{f^{n(k_i)}b\}_{i\in\omega}$ of $\{f^{n(k)} b\}_{k\in\omega}$. However (6) shows that

$$d(f(f^{n(k_i)}b), f^{n(k_i)}b)\to 0.$$

Hence $d(fu, u)=0$ and thus we have $fu=u$.

The unicity of the fixed point follows from the condition (5).

2. Family of mappings

THEOREM 2.1. *Let F be a non-empty family of mappings of a separated*

L-space $(M, \rightarrow)$ *which is d-complete for a nonnegative extended real valued function d on* $M\times M$. *Suppose that for every* $f, g\in F$, *there exists* $m=m(f, g)$, $n=n(f, g)$ *and* α, β *with* $0\leq\alpha<1$, $0<\beta\leq\infty$ *such that the following conditions are satisfied*

(7) $d(f^m x, g^n y)\leq\alpha\max\left\{d(x,y),\ d(f^m x, x),\ d(g^n y, y),\ \frac{1}{2}[d(f^m x, y)+d(g^n y, x)]\right\}$ *for every* $x, y\in M$ *with* $d(x,y)<\beta$.

(8) $d(f^m b, b)<\beta$ *for some* $b\in M$.

Then f has a fixed point and the sequence $\{f^n b\}_{n\in\omega}$ *converges to the fixed point, if in addition*

(9) $d(x,y)<\beta$ *for all fixed points* $x, y\in M$ *of f, then each* $f\in F$ *has a unique fixed point which is a unique common fixed point for F.*

PROOF. We shall first prove the theorem for the case when $m(f,g)=n(f,g)=1$ for every $f,g\in F$. By Theorem 1.1 each $f\in F$ has a unique fixed point $u_f\in M$. Since F is non-empty it will suffice to prove that $u_f=u_g$ for every $f,g\in F$. If $f,g\in F$, then we have

$$\begin{aligned} d(u_f, u_g) &= d(fu_f, gu_g) \\ &\leq\alpha\max\left\{d(u_f, u_g),\ \frac{1}{2}[d(u_g, fu_f)+d(u_f, gu_g)]\right\} \\ &\leq\alpha\max\{d(u_f, u_g),\ d(u_f, u_g)\} \\ &\leq\alpha\, d(u_f, u_g) \end{aligned}$$

which implies $u_f=u_g$. Now we shall prove the general case. Let f be in F. Then $f^{m(f,f)}$ has a unique fixed point v_f in M by what we have shown above. Hence we have

$$f^{m(f,f)}fv_f=f(f^{m(f,f)}v_f)=fv_f$$

which shows that $fv_f=v_f$. Since each fixed point of f is a fixed point of $f^{m(f,f)}$, it follows that v_f is the unique fixed point of f. If f,g are in F, then since

$$\begin{aligned} d(v_f, v_g) &= d(f^{m(f,g)}v_f, g^{m(f,g)}v_g) \\ &\leq\alpha\max\left\{d(v_f, v_g), \frac{1}{2}[d(v_f, v_g)+d(v_f, v_g)]\right\} \\ &\leq\alpha d(v_f, v_g) \end{aligned}$$

we have $v_f=v_g$. Therefore we have the desired conclusion since F is non empty.

THEOREM 2.2. *Let* $\{f_k\}_{k=1}^{n}$ *be a family of mappings of a separated L-space*

$(M, \to)$ into itself which is d-complete for a non-negative extended real valued function d on $M\times M$. If $\{f_k\}$ satisfied

(10) $$f_k f_l = f_l f_k \ (k, l=1, 2, \cdots, n)$$

(11) *for some α, β, $0<\alpha<1$, $0<\beta\leq\infty$ there is a system of positive integers m_1, $m_2, \cdots, m_n$ such that*

$$d(f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} x,\ f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} y) \leq \alpha \max\Big\{d(x,y),\ d(x,\ f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} x),\ d(y,\ f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} y),\ \frac{1}{2}[d(x, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} y) + d(y,\ f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} x)]\Big\}$$

$x, y\in M$ with $d(x, y)<\beta$.

(12) *$d(f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} b,\ b)<\beta$ for some $b\in M$.*

Then $\{f_k\}$ have a common fixed point and the sequence $\{f^n b\}_{n\in\omega}$ converges to the fixed point; if in addition

(13) *$d(x, y)<\beta$ for all fixed points $x, y\in M$ of $\{f_k\}$ then $\{f_k\}$ have a unique common fixed point.*

PROOF. Let $f=f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n}$. Then condition (11) takes the form

$$d(fx, fy)\leq\alpha \max\Big\{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{1}{2}[d(x,fy)+d(y,fx)]\Big\}$$

and hence by Theorem 1.1, f has a unique fixed point u in M. Therefore $fu=u$ and we have

$$f_k(fu)=f_k u \ (k=1, 2, \cdots, n).$$

By (10) we get $f(f_k u)=f_k u$. Since f has a unique fixed point u, we obtain $f_k u=u (k=1, \cdots, n)$. Hence u is a common fixed point of the family $\{f_k\}_{k=1}^{n}$. The unicity of the fixed point follows from the condition (13). This completes the proof of the theorem.

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