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SUBSTITUTION THEOREMS FOR THE TWO-DIMENSIONAL LAPLACE TRANSFORM I

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1. Introduction

Humbert in his works [1,2,3] developed the operational calculus of two variables based on the two-dimensional Laplace transform. Bose [4], Chakrabarty [5,6,7], Srivastava [8], Jain [9,10] and numerous other workers have contributed several papers on this same theme.

The two-dimensional Laplace transform [11] will be defined as

$$(1.1)\qquad f(p,q)=\int_0^\infty\int_0^\infty e^{-px-qy}\,F(x,y)\,dx\,dy,\ \mathrm{Re}(p)>0,\ \mathrm{Re}(q)>0,$$

in which $F(x,y)$ will be referred to as the original and it is a real or complex valued function of two real variables, defined on the region $R(0\le x<\infty,\ 0\le y<\infty)$ and is integrable in Lebesgue sense; $f(p,q)$ as the image, where p and q are complex parameters. The transform (1.1) will be symbolically denoted by

$$(1.2)\qquad f(p,q)\doteqdot F(x,y)\ \text{or}\ f(p,q)=L\{F\,;\,p,q\}.$$

The object at present is to obtain some formulae for the image of $K(x,y)F[g_1(x),\ g_2(y)]$ in terms of the image of $F(x,y)$ under (1.1) with certain restrictions on $F(x,y)$, $g_1(x)$ and $g_2(y)$. Such problems may be referred to as substitution theorems, since they involve the representation of the image of $K(x,y)F[g_1(x),\ g_2(y)]$ in terms of the image of $F(x,y)$. In Section 2, two substitution theorems are established and in the last section some examples of applications of these theorems are considered for specific functions $K(x,y)$, $g_1(x)$ and $g_2(y)$.

2. The first substitution theorem involves the representation of $L\{K(x,y)F[g_1(x),\ g_2(y)]\,;\,p,\ q\}$ in terms of $L\{F\,;\,p,q\}$. Since this representation will contain a double integral for which the convergence will depend upon $K(x,y)$, $g_1(x)$, $g_2(y)$ and $F(x,y)$, the method of attack will be to put on these three functions rather general conditions under which the desired representation is valid. For specific cases of $K(x,y)$, $g_1(x)$ and $g_2(y)$, further investigations can

be made concerning the allowable behaviour of $F(x,y)$. In addition for a given $g_1(x)$, $g_2(y)$ certain choices of $K(x,y)$ will produce simplification.

THEOREM 1. *If* (i) $K(x,y)$, $g_1(x)$, $g_2(y)$, $h_1=g_1^{-1}$ *and* $h_2=g_2^{-1}$ *are single-valued analytic functions* [12], *real on* $x>0$, $y>0$, *and such that* $g_i(0)=0$, $g_i(\infty)=\infty$ (*or* $g_i(0)=\infty$, $g_i(\infty)=0$) $i=1, 2$.

(ii) $f(p,q)=L\{F:p,q\}$ *which converges for* $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$,

(iii) *there exists a function* $\phi(p,q;u_1,u_2)$ *such that*

$$L\{\phi:s,t\}=\theta(p,q;s,t)$$

which converges for Re $(s)>0$, Re $(t)>0$ *and*

(2.1) $\theta(p,q;s,t)=e^{-ph_1(s)-qh_2(t)}\,K[h_1(s),h_2(t)]\;|h_1'(s)|\;|h_2'(t)|$:

(iv) *the integral*

(2.2) $\int_0^\infty\int_0^\infty\Big[\int_0^\infty\int_0^\infty e^{-su_1-tu_2}\,\phi(p,q;u_1,u_2)\,F(s,t)du_1du_2\Big]\,dsdt$ *converges absolutely*

for Re $(p)>a$, Re $(q)>b$: *then*

$$L\{K(x,y)\,F[g_1(x),\ g_2(y)]\ ;p,q\}$$
$$=\int_0^\infty\int_0^\infty\phi(p,q;u_1,u_2)\,f(u_1,u_2)\,du_1\,du_2$$

which converges for Re $(p)>a$, Re $(q)>b$.

PROOF. Since from (2.2) the iterated integral is absolutely convergent for Re $(p)>a$, Re $(q)>b$ and since

$$\int_0^\infty\int_0^\infty\Big|\int_0^\infty\int_0^\infty e^{-su_1-tu_2}\,\phi(p,q;u_1,u_2)\,du_1du_2\Big|\;\Big|F(s,t)\Big|\;dsdt$$
$$\leq\int_0^\infty\int_0^\infty\Big[\int_0^\infty\int_0^\infty\Big|e^{-su_1-tu_2}\,\phi(p,q;u_1,u_2)\,F(s,t)\Big|du_1du_2\Big]\,dsdt$$

then by the substitution from (2.1) the first iterated integral becomes

$$\int_0^\infty\int_0^\infty\Big|e^{-ph_1(s)-qh_2(t)}\,K[h_1(s),h_2(t)]\;|h_1'(s)|\;|h_2'(t)|\Big|\;|F(s,t)|\;dsdt.$$

Thus

$$\int_0^\infty\int_0^\infty e^{-ph_1(s)-qh_2(t)}\,K[h_1(s),h_2(t)]\,|h_1'(s)|\;|h_2'(t)|\;F(s,t)\;dsdt$$

is absolutely convergent for $\mathrm{Re}(p)>a$, $\mathrm{Re}(q)>b$.

There are two cases to be considered.

Case 1.

If $g_i(0)=0$ and $g_i(\infty)=\infty$, $i=1,2$, in view of (i), $h_1'(s)\geq 0$, $h_2(t)\geq 0$ so that if the substitutions $x=h_1(s)$, $y=h_2(t)$ are made, then

$$L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$$
$$=\int_0^\infty\int_0^\infty e^{-ph_1(s)-qh_2(t)}\,K[h_1(s),h_2(t)]\,|h_1'(s)|\ \ |h_2'(t)|\times F(s,t)\ ds\ dt.$$

Case 2.

If $g_i(0)=\infty$ and $g_i(\infty)=0$, $i=1,2$, then $h_1'(s)\leq 0$. $h_2'(t)\leq 0$ so that with the same substitution as in Case 1

$$L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$$
$$=\int_\infty^0\int_\infty^0 e^{-ph_1(s)-qh_2(t)}\,K[h_1(s),\ h_2(t)]\ |h_1'(s)|\ \ |h_2'(t)|\times F(s,t)\ ds\ dt,$$

which is equivalent to

$$L\{K(x,\ y)\ F[g_1(x),\ g_2(y)]\ :p,q\}$$
$$=\int_0^\infty\int_0^\infty e^{-ph_1(s)-qh_2(t)}\,K[h_1(s),\ h_2(t)]\ |h_1'(s)|\ \ |h_2'(t)|\times F(s,t)\ ds\ dt.$$

Thus in either case $L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$ is absolutely convergent for $\mathrm{Re}\ (p)>a$, $\mathrm{Re}(q)>\mathrm{b}$.

Now if the substitution from (2.1) is used in either of these cases, the result is

$$L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$$
$$=\int_0^\infty\int_0^\infty\Big[\int_0^\infty\int_0^\infty e^{-su_1-tu_2}\,\phi(p,q\ ;u_1,\ u_2)\ du_1\ du_2\Big]\ F(s,t)\ ds\ dt$$

which is absolutely convergent iterated integral. The order of integration can thus be changed so that

$$L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$$
$$=\int_0^\infty\int_0^\infty\Big[\int_0^\infty\int_0^\infty e^{-su_1-tu_2}\ F(s,t)\ dsdt\Big]\ \phi(p,q\ ;u_1,\ u_2)\ du_1du_2$$

and finally from (1,2)

$$L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$$

$$= \int_0^\infty \int_0^\infty \phi(p,q\,;u_1,u_2)\, f(u_1,u_2)\, du_1 du_2.$$

Four special cases of $K(x,y)$, three of which will simplify the image of $\phi(p,q\,;u_1,u_2)$ are worth noting at this stage.

(i) If $K(x,y)=e^{ax+by}$, then a form of translation theorem results, that is

$$L\{e^{ax+by}\,F[g_1(x),\,g_2(y)]\,;p,q\}$$
$$= \int_0^\infty \int_0^\infty \phi(p-a,\ q-b\,;u_1,u_2)\, f(u_1,u_2)\, du_1\, du_2.$$

Since the two dimensional Laplace transform is a linear transform the substitution of linear combinations of exponential functions for $K(x,y)$ will result in quite simple forms; for example, if $K(x,y)=\text{Sinh}\ (ax+by)$, then

$$L\{\text{Sinh}\ (ax+by)\ F[g_1(x),\,g_2(y)]\,;p,q\}$$
$$=\frac{1}{2}\int_0^\infty \int_0^\infty [\phi(p-a,\ q-b\,;u_1,u_2)-\phi(p+a,\ q+b\,;u_1,u_2)]\ f(u_1,u_2)\ du_1 du_2.$$

(ii) If $K(x,y)=|g_1'(x)|\ |g_2'(y)|$, then

$K[h_1(s),\ h_2(t)]=|h_1'(s)|^{-1}\ |h_2'(t)|^{-1}$ and $\theta(p,q\,;s,t)$ reduces to $e^{-ph_1(s)-qh_2(t)}$ that is

$$L\{\phi\,;s,t\}=\theta(p,q\,;s,t)=e^{-ph_1(s)-qh_2(t)}$$

(iii) If $K(x,y)=K[g_1(x),\,g_2(y)]$, then

$K[h_1(s),\ h_2(t)]=K[s,t]$ and

$$L\{\phi\,;s,t\}=e^{-ph_1(s)-qh_2(t)}\ K(s,t)|h_1'(s)|\ \ |h_2'(t)|.$$

In particular if $K(x,y)=[g_1(x),\,g_2(y)]^c$, then

$$L\{\phi\,;s,t\}=e^{-ph_1(s)-qh_2(t)}\ s^c\ t^c|h_1'(s)|\ \ |h_2'(t)|.$$

(iv) A combination of above cases (ii) and (iii) produces interesting forms of $\theta(p,q\,;s,t)$. In particular, if

$K(x,y)=|g_1'(x)|\ |g_2'(y)|K[g_1(x),g_2(y)]$, then $L\{\phi\,;s,t\}=K(s,t)e^{-ph_1(s)-qh_2(t)}$

or if $K(x,y)=|g_1'(x)|\ |g_2'(y)\ |\ [g_1(x).\ g_2(y)]^c$, then $L\{\phi\,;s,t\}=s^c\,t^c\,e^{-ph_1(s)--qh_2(t)}$

The next theorem will make use of the insertion of an arbitrary function which may be used to simplify the image of $\phi(p,q\,;u_1,u_2)$. This is done at the expense of complicating the representation so that the image of $F(x,y)$ itself no longer appears, but is replaced by the image of a product.

THEOREM 2. *If* (i) $A(x,y)$, $K(x,y)$, $g_1(x)$, $g_2(y)$, $h_1=g_1^{-1}$ *and* $h_2=g_2^{-1}$ *are single valued analytic functions* [11], *real on* $x>0$, $y>0$ *and such that* $g_i(0)=0$ *and* $g_i(\infty)=\infty$ (*or* $g_i(0)=\infty$ *and* $g_i(\infty)=0$) $i=1,2$;

(ii) $f^*(p,q)=L\{A(x,y)\ F(x,y) : p,q\}$ *which converges for* $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$;

(iii) *there exists a function* $\phi^*\ (p,q : u_1,u_2)$ *such that* $L\{\phi^* : s,t\}=\theta^*(p,q : s,t)$ *which converges for* $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$, *and*

$$\theta^*(p,q : s,t)=e^{-ph_1(s)-qh_2(t)}\ K[h_1(s),\ h_2(t)]\times|h_1'(s)|\,|h_2'(t)|\,[A(s,t)]^{-1}.$$

(iv) $\int_0^\infty\int_0^\infty\Big[\int_0^\infty\int_0^\infty e^{-su_1-tu_2}\ \phi^*(p,q : u_1,\ u_2)\ A(s,\ t)\ \ F(s,\ t)\ du_1\,du_2\Big]\ dsdt$ *converges absolutely for* $\mathrm{Re}\ (p)>a$, $\mathrm{Re}(q)>b$; *then*

$$L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$$
$$=\int_0^\infty\int_0^\infty \phi^*(p,\ q : u_1,\ u_2)\ f^*(u_1,u_2)du_1\,du_2$$

which converges for $\mathrm{Re}(p)>a$, $\mathrm{Re}(q)>b$.

The proof of Theorem 2 is similar to that of Theorem 1.

A special case for $A(x,y)=e^{ax+by}$ may be noted here $L\{e^{ax+by}\ F(x,y) : p,\ q\}=f(p-a,\ q-b)=f^*(p,q)$ and $e^{-as-bt}\ \theta(p,q ; s,t)=L\{G : p,q\}$,

where $G(p,q ; s,t)=\begin{cases}\phi(p,q : u_1-a,\ u_2-b)=\phi^*\ (p,q : u_1,u_2), & u_1>a,\ u_2>b\\ 0, & u_1<a,\ u_2<b,\end{cases}$

so that

$$L\{K(x,y)\ F[g_1(x),\ g_2(y)]\ ;p,q\}$$
$$=\int_a^\infty\int_b^\infty \phi(p,q : u_1-a,\ u_2-b)\ f(u_1-a,\ u_2-b)\ du_1du_2,$$

which could as well be obtained by substitution.

3. Applications

FORMULA 1. Let $g_1(x)=e^x-1$, $g_2(y)=e^y-1$ and $K(x,y)=(1-e^{-x}-e^{-y})^m$. From this $h_1(s)=\log_e(1+s)$ and $h_2(t)=\log_e(1+t)$. Hence

$|h_1'(s)|=(s+1)^{-1}$, $|h_2'(t)|=(t+1)^{-1}$ and $K[h_1(s),\ h_2(t)]=[1-(s+1)^{-1}$

$-(t+1)^{-1}]^m$, and

$$\theta(p,q:s,t)=\frac{1}{(s+1)^{p+1}(t+1)^{q+1}}\left[1-\frac{1}{s+1}-\frac{1}{t+1}\right]^m.$$

Now using the shift rule for an image and a known result [11, p. 139], we have

$$\phi(p,q:u_1,\ u_2)=\frac{(m!)^2 e^{-u_1-u_2} u_1^p u_2^q L_m^{p,q}(u_1,u_2)}{\Gamma(m+p+1)\ \Gamma(m+q+1)}$$

Therefore, for appropriate $F(x,y)$ satisfying the conditions of Theorem 1, we obtain

$$(3.1)\qquad L\{(1-e^{-x}-e^{-y})\ F(e^x-1,\ e^y-1):p,q\}$$
$$=\int_0^\infty\int_0^\infty \frac{(m!)^2 e^{-u_1-u_2} u_1^p u_2^q L_m^{p,q}(u_1,u_2)}{\Gamma(m+p+1)\ \Gamma(m+q+1)} f(u_1,\ u_2)\ du_1\,du_2.$$

FORMULA 2. Let $g_1(x)$ and $g_2(y)$ be the same as in Formula 1 and $K(x,y) = e^{e^{-x-y}}$ From these

$$\theta(p,q:s,t)=\frac{1}{(s+1)^{p+1}(t+1)^{q+1}}\ e^{-\frac{1}{(s+1)(t+1)}}$$

Now using the shift rule for an image and a known result [11, p. 145], we have

$$\phi(p,q:u_1,\ u_2)=e^{-u_1-u_2}\, u_1^{\frac{1}{3}(2p-q)}\ u_2^{\frac{1}{3}(2q-p)}\ J_{p,q}\ (3\sqrt[3]{u_1,\ u_2}\,)$$

Therefore, for appropriate $F(x,y)$ satisfying the conditions of Theorem 1, we get

$$(3.2)\quad L\{e^{-e^{-x-y}}.\ F(e^x-1,\ e^y-1):p,q\}$$
$$=\int_0^\infty\int_0^\infty e^{-u_1-u_2}\, u_1^{\frac{1}{3}(2p-q)}\ u_2^{\frac{1}{3}(2q-p)}\ J_{p,q}(3\sqrt[3]{u_1u_2}\,)\ f(u_1,u_2)\ du_1du_2.$$

FORMULA 3. Let $g_1(x)=x$, $g_2(y)=y$ and $K(x,y)=1$. From this $\theta(p,q:s,t) =e^{-ps-qt}$ and no $\phi(p,q:u_1,u_2)$ exists which has this function as an image. Let

$$A(s,t)=\frac{(st+q)^p}{e^{ps+qt}},\ \text{Re}(p)>0,\ \text{Re}(q)>0$$

and apply Theorem 2 so that

$$\theta^*(p,q:s,t)=\frac{1}{(st+q)^p},\ \text{Re}(p)>0.$$

Now using a known result [11, p. 135], we have

$$\phi^*(p, q : u_1, u_2) = \frac{1}{\Gamma(p)} \left(\frac{u_1 u_2}{q}\right)^{\frac{1}{2}(p-1)} J_{p-1}(2\sqrt{q u_1 u_2}) .$$

Therefore, for appropriate $F(x, y)$ satisfying conditions of Theorem 2, we obtain

(3.3) $$L\{F(x, y) : p, q\}$$

$$= \int_0^\infty \int_0^\infty \frac{1}{\Gamma(p)} \left(\frac{u_1 u_2}{q}\right)^{\frac{1}{2}(p-1)} \cdot J_{p-1}(2\sqrt{q u_1 u_2})\, f^*(u_1, u_2)\, du_1 du_2.$$

FORMULA 4. Let $g_1(x)$ and $g_2(y)$ be the same as in Formula 3 and $K(x, y) = yx^{-n}$. From this $\theta(p, q : s, t) = ts^{-n} e^{-ps-qt}$ and no $\phi(p, q : u_1, u_2)$ exists which has this function as image. Let

$$A(s, t) = \frac{t(st+q)^p}{e^{ps+qt}}, \ \mathrm{Re}(p)>0, \ \mathrm{Re}(q)>0$$

and apply Theorem 2 so that

$$\phi^*(p, q : s, t) = \frac{1}{s^n (st+q)^p}, \ \mathrm{Re}(p)>0.$$

Now using a known result [11, p. 137], we have

$$\phi^*(p, q : u_1, u_2) = \frac{(u_2)^{p-1}}{\Gamma(p)} \left(\frac{u_1}{q u_2}\right)^{\frac{p+n-1}{2}} \cdot J_{p+n-1}(2\sqrt{q u_1 u_2}).$$

Therefore, for appropriate $F(x, y)$ satisfying conditions of Theorem 2, we have

(3.4) $$L\{yx^{-n} F(x, y) : p, q\}$$

$$= \int_0^\infty \int_0^\infty \frac{(u_2)^{p-1}}{\Gamma(p)} \cdot \left(\frac{u_1}{q u_2}\right)^{\frac{p+n-1}{2}} \cdot J_{p+n-1}(2\sqrt{q u_1 u_2}) \times f^*(u_1, u_2)\, du_1 du_2$$

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