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MULTILATERAL GENERATING FUNCTIONS FOR THE PRODUCTS OF HYPERGEOMETRIC POLYNOMIALS AND SEVERAL GENERALIZED HYPERGEOMETRIC FUNCTIONS

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1. Introduction

By exploiting the properties of the differential operator $\lambda x^k + x^{k+1}(d/dx)$, Patil and Thakare [8] obtained multilateral generating functions involving both the Konhauser Biorthogonal polynomial sets. By resorting to the same technique the authors [7] recently obtained multilateral generating functions involving the products of polynomials of Srivastava and Singhal [14] and several generalized hypergeometric functions.

In this paper we obtain multilateral generating function (2.1) given below by adopting series manipulation technique and Gauss transformation for hypergeometric functions. Our result is quite general in the sense that it is possible for us to obtain multilinear generating functions involving the products of several known polynomials including the classical orthogonal polynomials.

The detailed discussion is postponed to appropriate sections.

It may be mentioned that the main result obtained earlier by the authors [7] is different from the one obtained here.

(2) For convenience, we shall write $H(y_1, y_2, \cdots, y_k)$ for

$$\prod_{j=1}^{k} {}_{m_j+B^{(j)}}F_{c}(j) \left[\begin{matrix} \Delta(m_j, -n_j), & (b^{(j)}); \\ & (c^{(j)}); \end{matrix} \left(-\frac{m_j}{f_j(y_j)}\right)^{m_j} \phi_j(y_j)\right]$$

where $f_j(x)$ and $\phi_j(x)$ are nonzero real functions, m_js are positive integers, δ is a complex parameter, $\Delta(\beta,\alpha)$ stands for the set of β parameters $\frac{\alpha}{\beta}, \frac{\alpha+1}{\beta}, \cdots, \frac{\alpha+\beta-1}{\beta}$ $(\beta\geq 1)$, $(b^{(j)})$ denotes the sequence of $B^{(j)}$, complex parameters $b_1^{(j)}, b_2^{(j)}, \cdots, b_B^{(j)}(j)$ independent of n_j's with similar interpretation for $(c^{(j)})$, $(j=1, 2, \cdots, k)$.

2. Main result

$$(2.1) \qquad \sum_{n_1, \cdots, n_k=0}^{\infty} (\delta)_{m+\Sigma n_i}\ {}_2F_1\left[\begin{matrix} -m-\Sigma n_i, & a; \\ & \delta: \end{matrix}\ x\right] H(y_1, \cdots, y_k)$$

$$\cdot \frac{(f_1(y_1)u_1)^{n_1}}{n_1\,!}, \cdots, \frac{(f_k(y_k)u_k)^{n_k}}{n_k\,!}$$

$$=(1-x)^{-a}(\delta)_m\, \theta_k^{-\delta-m}\, F^{1:1;B^1;\cdots;B^{(k)}}_{0:1;C^1;\cdots;C^{(k)}}\left(\begin{matrix}[\delta+m:1,m_1,\cdots m_k]:[a:1]:[(b^1):1];\cdots;[(b^{(k)}:1];\\ \overline{\qquad\qquad\qquad}:[\delta:1]:[(c^1):1];\cdots;[(c^{(k)}:1];\end{matrix}\right.$$

$$\left.\frac{x}{(x-1)\theta_k}, \phi_1(y_1)\left(\frac{u_1}{\theta_k}\right)^{m_1}, \cdots, \phi_k(y_k)\left(\frac{u_k}{\theta_k}\right)^{m_k}\right)$$

where,

(2.2) $\theta_k=1-\sum_{j=1}^{k} f_j(y_j)u_j$ $(k=1, 2, \cdots)$, m is some fixed integer; and on the right hand we use the notation for the generalized Lauricella function of several variables defined and studied by Srivastava and Daoust [13; p. 454]; see also Exton [4; p. 109].

PROOF. Firstly let us consider the sum

$$\sum_{n_1,\cdots,n_k=0}^{\infty} (\lambda)_{m+\Sigma n_i}\, H(y_1, \cdots, y_k) \frac{(f_1(y_1)u_1)^{n_1}}{n_1\,!} \cdots \frac{(f_k(y_k)u_k)^{n_k}}{n_k\,!}$$

$$=\sum_{n_1,\cdots,n_k=0}^{\infty} (\lambda)_{m+\Sigma n_i} \prod_{j=1}^{k} \sum_{r_j=0}^{[n_j/m_j]} \frac{[(b^{(j)})]_{r_j}(-1)^{m_j r_j}(-n)_{m_j r_j}}{[(c^{(j)})]_{r_j}\, n_j\,!\; r_j\,!} (j_j(y_j))^{n_j-m_j r_j}$$

$$\cdot (\phi_j(y_j))^{r_j}\, u_j^{n_j}$$

where $[(b^{(j)})]_{n_j}$ has the interpretation $\prod_{l=1}^{B^{(j)}} [b_l^{(j)}]_{n_j}$ and so also for $[(c^{(j)})]_{n_j}$.

By direct computation the above expression reduces to

$$(\lambda)_m \sum_{r_1,\cdots,r_k=0}^{\infty} (\lambda+m)_{\Sigma m_i r_i} \prod_{j=1}^{k} \left\{ \frac{[(b^{(j)})]_{r_j}}{[(c^{(j)})]_{r_j}} \frac{(\phi_j(y_j)u_j^{m_j})^{r_j}}{r_j\,!} \right\}$$

$$\cdot \left\{ \sum_{n_1,\cdots,n_k=0}^{\infty} (\lambda+m+\Sigma m_i r_i)_{\Sigma n_i} \prod_{j=1}^{k} \frac{(f_j(y_j)u_j)^{n_j}}{n_j\,!} \right\}$$

$$=(\lambda)_m \sum_{r_1,\cdots,r_k=0}^{\infty} (\lambda+m)_{\Sigma m_i r_i} \left[1-\sum_{l=1}^{k} f_l(y_l)u_l\right]^{-\lambda-m-\Sigma m_i r_i}$$

$$\cdot \prod_{j=1}^{k} \frac{[(b^{(j)})]r_j(\phi_j(y_j)u_j^{m_j})^{r_j}}{[(c^{(j)})]_{r_j}\, r_j\,!}.$$

Thus we obtain

$$(2.3)\quad \sum_{n_1,\cdots,n_k=0}^{\infty}(\lambda)_{m+\Sigma n_i}H(y_1,\cdots,y_k)\frac{(f_1(y_1)u_1)^{n_1}}{n_1!}\cdots\frac{(f_k(y_k)u_k)^{n_k}}{n_k!}$$

$$=(\lambda)_m\theta_k^{-\lambda-m}\,F^{1:B^1;\cdots;B^{(k)}}_{0:C^1;\cdots;C^{(k)}}\left(\begin{matrix}[\lambda+m:m_1,\cdots,m_k]:[(b^1):1];\cdots;[(b^{(k)}):1];\\ \overline{\qquad\qquad\qquad}:[(c^1):1];\cdots;](c^{(k)}):1];\end{matrix}\right.$$

$$\left.\phi_1(y_1)\left(\frac{u_1}{\theta_k}\right)^{m_1}\cdots\phi_k(y_k)\left(\frac{u_k}{\theta_k}\right)^{m_k}\right),$$

where θ_k is given by (2.2).

Let S denote the left hand side of (2.1). By often used Euler transformation, one has,

$$S=(1-x)^{-a}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}(-x/(1-x))^n\sum_{n_1,\cdots,n_k=0}^{\infty}(\delta+n)_{m+\Sigma n_i}$$

$$\cdot H(y_1,\cdots,y_k)\frac{(f_1(y_1)u_1)^{n_1}}{n_1!}\cdots\frac{(f_k(y_k)u_k)^{n_k}}{n_k!}$$

Using (2.3) and writing the expansion of the generalized Lauricella function involved, we obtain

$$S=(1-x)^{-a}\sum_{n,n_1,\cdots,n_k=0}^{\infty}\frac{(a)_n(x/(x-1))^n}{n!}(\delta+n)_m\theta_k^{-\delta-n-m}$$

$$\cdot(\delta+n+m)_{\Sigma m_i n_i}\prod_{j=1}^{k}\frac{[(b^{(j)})]_{n_j}}{[(c^{(j)})]_{n_j}n_j!}\left[\phi_j(y_j)\left(\frac{u_j}{\theta_k}\right)^{m_j}\right]^{n_j}$$

where θ_k is given by (2.2). This can be further simplified to obtain final result (2.1).

It may be mentioned that our result generalizes the consideration of Thakare and Karande [17].

3. Applications

(1) If we select $f_j(y_j)=m_j=1$; and $\phi_j(y_j)=-y_j$ for each $j=1, 2, \cdots, k$, we shall obtain the result (24) of Srivastava and Singhal [15, p. 1244].

It may be mentioned that Srivastava and Singhal [15] obtain their result by indicating the use of two techniques; first being the use of Laplace transform and its inverse, and the second being the employment of differential operators.

(2) By specializing the parameters it is fairly easy to obtain a large number of known or new multilateral/multilinear generating relations involving generalized Rice polynomials, Jacobi polynomials, Laguerre polynomials, Hermite

polynomials, and their generalizations due to Brafman [1]; Gould-Hopper [5] Lahiri [6]; and Bragg [2]. But we shall not state them here as most of them are explicitly mentioned in earlier work of the authors [7]. However we shall indicate very briefly some results which seem to be new and interesting.

By specializing the parameters one can obtain a formula containing the product of hypergeometric polynomial and several Brafman polynomials [1] which are given by

$$f_n\left[k;\ \begin{matrix}\alpha_1,\cdots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{matrix}\ x\right] = {}_{k+p}F_q\left[\begin{matrix}\Delta(k,-n),\ \alpha_1,\cdots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{matrix}\ x\right].$$

Recall Gould-Hopper generalizations of the Hermite polynomials [5, p.58] given by

$$g_n^r(x,h) = \sum_{k=0}^{[n/r]} \frac{n!}{k!\,(n-rk)!}\, h^k\, x^{n-rk}$$

$$= x^n\ {}_rF_0[\triangle(r,-n);\ -;\ h(-r/x)^r],\quad (r \geq 1).$$

These are a particular case of Brafman polynomials.

We have, by selecting $B^{(j)} = C^{(j)} = 0$, $f_j(y_j) = y_j$ and $\phi_j(y_j) = h_j$, $(j=1, 2, \cdots, k)$ in (2.1) the following multilateral generating function,

$$(3.1)\quad \sum_{n_1,\cdots,n_k=0}^{\infty} (\delta)_{m+\Sigma n_i}\ {}_2F_1\left[\begin{matrix}-m-\Sigma n_i,\ a;\\ \delta;\end{matrix}\ x\right] \prod_{j=1}^{k} g_{n_j}^{m_j}(y_j, h_j)\ \frac{u_j^{n_j}}{n_j!}$$

$$= (1-x)^{-a}(\delta)_m \triangle_k^{-\delta-m}\ F_{0:1;0;\cdots;0}^{1:1;0;\cdots;0}\left(\begin{matrix}[\delta+m:1,\ m_1,\cdots,m_k] : [a:1];\text{———};\\ \text{———} : [\delta:1];\text{———};\end{matrix}\right.$$

$$\left.\frac{x}{(x-1)\triangle_k},\ h_1\left(\frac{u_1}{\triangle_k}\right)^{m_1},\ \cdots,\ h_k\left(\frac{u_k}{\triangle_k}\right)^{m_k}\right);$$

where

$$(3.2)\qquad \triangle_k = 1 - \sum_{j=1}^{k} y_j u_j,\quad (k=1, 2, \cdots).$$

Srivastava [16] pointed out that the polynomials $H_{n,m,\nu}(x)$ defined by Lahiri [6] and $g_n^m(x)$ defined by Bragg [2] are particular cases of $g_n^r(x,h)$ by showing,

$$H_{n,m,\nu}(x) = \nu^n\ g_n^m(x,-1) = g_n^m(\nu x,-1);\ \text{and}\ g_n^m(x) = g_n^m(mx,-1).$$

Hence from (3.1) one can obtain corresponding formulas for the polynomials $H_{n,m,\nu}(x)$ and $g_n^m(x)$.

For Gegenbauer polynomials [9; p.280], we have

$$C_n^{\nu}(x)=\frac{(2\nu)_n}{n\,!}x^n\ {}_2F_1\Big[\triangle(2,-n);\ \nu+1/2;\ \frac{x^2-1}{x^2}\Big]$$

Hence (2.1) yields as a special case

$$(3.3)\quad \sum_{n_1,\cdots,n_k=0}^{\infty}(\delta)_{m+\Sigma n_i}\ {}_2F_1\Big[{-m-\Sigma n_i,\ a;\atop \delta;}x\Big]\prod_{j=1}^{k}\frac{C_{n_j}^{\nu_j}(y_j)}{(2\nu_j)n_j}u_j^{n_j}$$
$$=(1-x)^{-a}(\delta)_m\triangle_k^{-\delta-m}(k)_{H_4}^{(k+1)}\Big(\delta+m,a;\ \nu_1+\frac{1}{2},\ \cdots,\ \nu_k+\frac{1}{2},\ \delta;$$
$$\frac{(y_1^2-1)u_1^2}{4\triangle_k^2},\cdots,\frac{(y_k^2-1)\,u_k^2}{4\triangle_k^2},\ \frac{x}{(x-1)\triangle_k}\Big),$$

where $\triangle_k$ is given by (3.2); and ${}^{(k)}H_4^{(n)}$ denotes a generalization of the Horn function H_4; see Exton [4, p.97].

In particular one can have a bilateral generating function in view of the relationship [9, p.254 eq.1] in the form;

$$(3.4)\quad \sum_{n=0}^{\infty}(m+n)!P_{m+n}^{(\alpha,\beta-m-n)}(x)\ \frac{C_n^{\nu}(y)}{(2\nu)_n}t^n$$
$$=2^{1+\alpha+\beta}\,(x+1)^{-1-\alpha-\beta}(1+\alpha)_m(1-ty)^{-1-\alpha-m}$$
$$\cdot H_4\Big(1+\alpha+m,\ 1+\alpha+\beta;\ \nu+\frac{1}{2},\ 1+\alpha;\ \frac{(y^2-1)t^2}{4(1-ty)^2},\ \frac{(x-1)}{(x+1)(1-ty)}\Big)$$

where H_4 is Horn's function; see Exton [4, p.36]. By choosing $1+\alpha=\nu$ and taking the limit as $x\to 1$ one can easily obtain a generating relation (8) of [9, p.279]. Also by using the result (7) p.255 of [9] in (3.3) one can get,

$$(3.5)\quad \sum_{n=0}^{\infty}(m+n)!\ P_{m+n}^{(\alpha-m-n,\beta-m-n)}\ (x)\frac{C_n^{\nu}(y)}{(2\nu)_n}\,t^n$$
$$=(-1)^m(-\alpha-\beta)_m\Big(\frac{x+1}{2}\Big)\Big(\frac{x-1}{2}\Big)^{m-a}\Big[1+\frac{1}{2}(x-1)yt\Big]^{\alpha+\beta-m}$$
$$\cdot H_4\Big(-\alpha-\beta+m,\ -\alpha;\ \nu+\frac{1}{2},\ -\alpha-\beta;\frac{(y^2-1)(1-x)^2t^2}{4[2-(1-x)yt)]^2},\ \frac{4}{(x+1)[2-(1-x)yt]}\Big).$$

From (2.1) one can deduce,

$$(3.6)\quad \sum_{n_1,\cdots,n_k=0}^{\infty}(\delta)_{m+\Sigma n_i}\ {}_2F_1\Big[{-m-\Sigma n_i\ a;\atop \delta;}\ x\Big]\prod_{j=1}^{r}\frac{C_{n_j}^{\nu_j}(y_j)}{(2\nu_j)_{n_j}}u_j^{n_j}$$

$$\prod_{l=r+1}^{k} {}_2F_1\left[\begin{smallmatrix}-n_l & b_l; \\ & c_l; \end{smallmatrix} y_l\right]\frac{u_l^{n_l}}{n_l!}$$

$$=(1-x)^{-a}(\delta)_m \mu_k^{-\delta-m} {}^{(r)}H_4^{(k+1)}\Big((\delta+m, b_{r+1}, \cdots, b_k, a;$$

$$\nu_1+\frac{1}{2}, \cdots, \nu_r+\frac{1}{2}, c_{r+1}, \cdots, c_k, \delta; \frac{(y_1^2-1)u_1^2}{4\mu_k^2}, \cdots, \frac{(y_r^2-1)u_r^2}{4\mu_k^2},$$

$$\frac{-y_{r+1}u_{r+1}}{\mu_k}, \cdots, \frac{-y_k u_k}{\mu_k}, \frac{x}{(x-1)\mu_k}\Big),$$

where

(3.7) $$\mu_k=1-\sum_{j=1}^{r} y_j u_j-\sum_{l=r+1}^{k} u_l, \quad (k=1, 2, \cdots).$$

In particular by using the result-1 of [9, p. 254], (3.6) yields,

(3.8) $$\sum_{n_1, \cdots, n_k=0}^{\infty} (m+\Sigma n_i)!\ P_{m+\Sigma n_i}^{(\alpha, \beta-m-\Sigma n_i)}(x) \prod_{j=1}^{r} \frac{C_{n_j}^{\nu_j}(y_j)}{(2\nu_j)_{n_j}} u_j^{n_j}$$

$$\cdot \prod_{l=r+1}^{k} \frac{P_{n_l}^{(\alpha_l, \beta_l-n_l)}(y_l)}{(1+\alpha_l)_{n_l}} u_l^{n_l}$$

$$=2^{1+\alpha+\beta}(1+x)^{-1-\alpha-\beta}(1+\alpha)_m \mu_k^{-1-\alpha-m} {}^{(r)}H_4^{(k+1)}\Big(1+\alpha+m, 1+\alpha_{r+1}$$

$$+\beta_{r+1}, \cdots, 1+\alpha_k+\beta_k, 1+\alpha+\beta; \nu_1+\frac{1}{2}, \cdots, \nu_r+\frac{1}{2}, 1+\alpha_{r+1}, \cdots, 1+\alpha_k,$$

$$1+\alpha; \frac{(y_1^2-1)u_1^2}{4\mu_k^2}, \cdots, \frac{(y_r^2-1)u_r^2}{4\mu_k^2}, \frac{(y_{r+1}-1)u_{r+1}}{2\mu_k},$$

$$\cdots, \frac{(y_k-1)u_k}{2\mu_k}, \frac{x-1}{(x+1)\mu_k}\Big),$$

where μ_k is as given by (3.7).

By using earlier mentioned result of Rainville [9, p. 255], we have from (3.6)

(3.9) $$\sum_{n_1, \cdots, n_k=0}^{\infty} (m+\Sigma n_i)!\ P_{m+\Sigma n_i}^{(\alpha-m-\Sigma n_i, \beta-m-\Sigma n_i)}(x) \prod_{j=1}^{r} \frac{C_{n_j}^{\nu_j}(y_j)}{(2\nu_j)_{n_j}} u_j^{n_j}$$

$$\cdot \prod_{l=r+1}^{k} \frac{P_{n_l}^{(\alpha_l-n_l, \beta_l-n_l)}(y_l)}{(-\alpha_l-\beta_l)_{n_l}} u_l^{n_l}$$

$$=(-1)^m(-\alpha-\beta)_m\Big(\frac{x+1}{2}\Big)^{\alpha}\Big(\frac{x-1}{2}\Big)^{m-\alpha} \lambda_k^{\alpha+\beta-m} {}^{(r)}H_4^{(k+1)}\Big(-\alpha-\beta+m, -\alpha_{r+1},$$

$$\cdots, -\alpha_k, -\alpha; \nu_1+\frac{1}{2}, \cdots, \nu_r+\frac{1}{2}, -\alpha_{r+1}-\beta_{r+1}, \cdots, -\alpha_k-\beta_k,$$

$$-\alpha-\beta; \frac{(y_1^2-1)(1-x)^2u_r^2}{16\lambda_k^2}, \cdots, \frac{(y_r^2-1)(1-x)^2u_r^2}{16\lambda_k^2}, \frac{(x-1)u_{r+1}}{2\lambda_k},$$

$$\cdots, \frac{(x-1)u_k}{2\lambda_k}, \frac{2}{(x+1)\lambda_k}\Big),$$

where $\lambda_k=1-\frac{1}{2}(1-x)\sum_{j=1}^{r} y_ju_j-\frac{1}{4}(x-1)\sum_{l=r+1}^{k}(y_l-1)u_l$, $(k=1, 2, \cdots)$.

In (3.6) replace x by $\frac{x}{a}$, each y_l by $\frac{y_l}{b_l}$, δ by $1+\alpha$, each c_l by $1+\alpha_l$ and taking limits as $a\to\infty$ and each $b_l\to\infty$ one obtains

$$(3.10)\quad \sum_{n_1,\cdots,n_k=0}^{\infty}(m+\Sigma n_i)!\, L_{m+\Sigma n_i}^{(\alpha)}(x)\prod_{j=1}^{r}\frac{C_{n_j}^{\nu_j}(y_j)u_j^{n_j}}{(2\nu_j)_{n_j}}$$

$$\cdot\prod_{l=r+1}^{k}L_{n_l}^{(\alpha_l)}(y_l)u_l^{n_l}/(1+\alpha_l)_{n_l}$$

$$=e^x(1+\alpha)_m\mu_k^{-1-\alpha-m}\,{}^{(r)}H_4^{*(k+1)}\Big(1+\alpha+m; \nu_1+\frac{1}{2}, \cdots, \nu_r+\frac{1}{2},$$

$$1+\alpha_{r+1}, \cdots, 1+\alpha_k, 1+\alpha; \frac{(y_1^2-1)u_1^2}{4\mu_k^2}, \cdots, \frac{(y_r^2-1)u_r^2}{4\mu_k^2}, \frac{-y_{r+1}u_{r+1}}{\mu_k}$$

$$\cdots, \frac{-y_ku_k}{\mu_k}, \frac{-x}{\mu_k}\Big),$$

where μ_k is given by (3.7); and

$${}^{(k)}H_4^{*(n)}(a;c_1, \cdots, c_n; x_1, \cdots, x_n)$$

$$=\lim_{\varepsilon\to\infty}{}^{(k)}H_4^{(n)}\Big(a, \varepsilon, \cdots, \varepsilon; c_1, \cdots, c_n; x_1, \cdots, x_k, \frac{x_{k+1}}{\varepsilon}, \cdots, \frac{x_n}{\varepsilon}\Big).$$

In fact the relations (3.4) and (3.5), are essentially equivalent and so also (3.8) and (3.9), due to fairly known result,

$$P_n^{(\alpha,\beta-n)}(x)=\Big(\frac{1-x}{2}\Big)^n P_n^{(-\alpha-\beta-n-1,\beta-n)}\Big(\frac{x+3}{x-1}\Big), \text{ for each integer } n\geq 0.$$

Also it should be noted that since,

$L_n^{(\alpha)}(x)=\lim_{\beta\to 0}P_n^{\left(\alpha,\frac{1}{\beta}\right)}(1-2x\beta)$, one can derive formula (3.10) as a limiting case of formula (3.8) or (3.9). Results (3.4) and (3.5) are respectively contained in Results (3.8) and (3.9); see Exton [4, p.98] for necessary formulas needed for such reduction. Also one can easily obtain respectively the formulas (5), (29) and (30) of Srivastava and Singhal [15] from relations (3.10), (3.8) and (3.9) by recalling that,

$$(0)H_4^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n) = F_A^{(n)}(a, b_1, \cdots, b_n; c_1, \cdots, c_n; x_1, \cdots, x_n);$$

see Exton [4, p.98].

4. Other applications

Select $m_j=1$ $(j=1, \cdots, k)$; and apply the result (2.4-4) of [3, p.28] to obtain,

$$(4.1)\quad \sum_{n_1, \cdots, n_k=0}^{\infty} (\delta)_{m+\Sigma n_i}\, {}_2F_1\left[\begin{matrix} -m-\Sigma n_i, & a; \\ & \delta; \end{matrix}\; x\right] \prod_{j=1}^{k} \frac{[(b^{(j)})]_{n_j}}{[(c^{(j)})]_{n_j}} \frac{u_j^{n_j}}{n_j!}$$

$$\cdot\, {}_{1+C^{(j)}}F_{B^{(j)}}\left[\begin{matrix} -n_j, & (1-(c^{(j)})-n_j); \\ & (1-(b^{(j)})-n_j); \end{matrix}\; (-1)^{1+B^{(j)}+C^{(j)}} \frac{f_j(y_j)}{\phi_j(y_j)}\right]$$

$$=(1-x)^{-a}(\delta)_m \theta_k^{-\delta-m} F^{1:1;B^1;\cdots;B^{(k)}}_{0:1;C^1;\cdots;C^{(k)}}\left(\begin{matrix} [\delta+m:1, \cdots, 1] : [a:1]; \\ \rule{4em}{0.4pt} : [\delta:1]; \end{matrix}\right.$$

$$\left.\begin{matrix} [(b^1):1];\cdots;[(b^{(k)}):1]; \\ [(c^1):1];\cdots;[(c^{(k)}):1]; \end{matrix}\; \frac{x}{(x-1)\theta_k}, \frac{u_1}{\theta_k}, \cdots, \frac{u_k}{\theta_k}\right),$$

where $\theta=1-\sum_{j=1}^{k} \frac{f_j(y_j)}{\phi_j(y_j)} u_j$, $(k=1, 2, \cdots)$, and

$(1-(b^{(j)})-n_j)$ represents the sequence of $B^{(j)}$ parameters $1-b_1^{(j)}-n_j, \cdots,$ $1-b_{B^{(j)}}^{(j)}-n_j$ with similar interpretation for $(1-(c^{(j)})-n_j)$.
When we allow each $\phi_j(y_j)\longrightarrow\infty$, we obtain

$$(4.2)\quad \sum_{n_1, \cdots, n_k=0}^{\infty} (\delta)_{m+\Sigma n_i} \left\{\prod_{j=1}^{k} \frac{[(b^{(j)})]_{n_j} u_j^{n_j}}{[(c^{(j)})]_{n_j} n_j!}\right\} {}_2F_1\left[\begin{matrix} -m-\Sigma n_i, & a; \\ & \delta; \end{matrix}\; x\right]$$

$$=(1-x)^{-a}(\delta)_m F^{1:1;B^1;\cdots;B^{(k)}}_{0:1;C^1;\cdots;C^{(k)}}\left(\begin{matrix} [\delta+m:1, \cdots, 1] : [a:1]; \\ \rule{4em}{0.4pt} : [\delta:1]; \end{matrix}\right.$$

$$\left.\begin{matrix} [(b^1):1];\cdots;[(b^{(k)}):1]; \\ [(c^1):1];\cdots;[(c^{(k)}):1]; \end{matrix}\; \frac{x}{x-1}, u_1, \cdots, u_k\right).$$

Relation (4.2) contains the result (4.1) of Saran [10, p.786].

If summations involving $n_2, \cdots, n_k$ are absent, and $x\to 0$ in (2.1) we shall obtain a linear function

$$(4.3)\quad \sum_{n=0}^{\infty}(\delta)_{m+n}\ {}_{p+q}F_r\left[\begin{matrix}\Delta(p,-n),\ \alpha_1,\cdots,\alpha_q;\\ \beta_1,\cdots,\beta_r;\end{matrix}\left(\frac{-p}{f(y)}\right)^p\phi(y)\right]\frac{[f(y)t]^n}{n!}$$
$$=[1-f(y)t]^{-\delta-m}(\delta)_m\ {}_{p+q}F_r\left[\begin{matrix}\Delta(p,\delta+m),\ \alpha_1,\cdots,\alpha_q;\\ \beta_1,\cdots,\beta_r;\end{matrix}\left(\frac{pt}{1-f(y)t}\right)^p\phi(y)\right].$$

It may be seen that (4.3) easily follows from (2.3). Above relation is given in Srivastava [11, p.66]; see also Srivastava [12, p.203].

One can obtain by direct computation a further generalization of above result in the form,

$$(4.4)\quad \sum_{n=0}^{\infty}\varepsilon_n\ {}_{p+q}F_r\left[\begin{matrix}\Delta(p,-n),\ \alpha_1,\cdots,\alpha_q;\\ \beta_1,\cdots,\beta_r;\end{matrix}\left(\frac{-p}{f(x)}\right)^p\phi(x)\right]\frac{[f(x)t]^n}{n!}$$
$$=\sum_{k=0}^{\infty}\varepsilon_{pk}\ {}_{q+i}F_{r+j}\left[\begin{matrix}\alpha_1,\cdots,\alpha_q,\ a_1+pk,\cdots,a_i+pk;\\ \beta_1,\cdots,\beta_r,\ b_1+pk,\cdots,b_j+pk;\end{matrix}f(x)t\right]\frac{(\phi(x)t^p)^k}{k!}$$

$$\text{where } \varepsilon_n=\frac{(a_1)_n,\cdots,(a_i)_n}{(b_1)_n,\cdots,(b_j)_n},\quad (n\geq 0).$$

Incidently (4.4) contains the result (3.3) of Srivastava [16, p.460]. Relation (4.4) is also to be found in Srivastava [11, p.67].

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