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A NEW CLOSURE OPERATOR FOR NON-T_1 TOPOLOGIES

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1. Introduction

The concept of a generalized closed (g-closed) subset of a topological space was introduced by Norman Levine in [6] and has been discussed in papers appearing in this, and other, journals (see [1], [4], and [5]). In the short note which follows, we shall use the g-closed subsets of a space $(X, \mathscr{T})$ to define a new closure operator, and thus a new topology $\mathscr{T}^*$, on the space and shall examine some of the properties of this new topology, with emphasis on the transfer of "regularity" conditions on $(X, \mathscr{T})$ to "separation" conditions on $(X, \mathscr{T}^*)$.

2. Preliminaries

DEFINITION 2.1 (Levine [6]). In a topological space $(X, \mathscr{T})$, a subset A is *g-closed* if $c(A) \subseteq O$ whenever $A \subseteq O \in \mathscr{T}$ (here c denotes the closure operator in $(X, \mathscr{T})$).

THEOREM 2.2. *For each $x \in X$, either $\{x\}$ is closed or $\mathscr{C}\{x\}$ is g-closed ($\mathscr{C}$ denotes the complement operator).*

PROOF. If $\{x\}$ is not closed, then the only open superset of $\mathscr{C}\{x\}$ is X itself. Thus the closure of $\mathscr{C}\{x\}$ is contained in each of its neighborhoods and $\mathscr{C}\{x\}$ is g-closed.

THEOREM 2.3 (Levine [6]). *The union of two g-closed sets is g-closed.*

PROOF. The proof is immediate.

DEFINITION 2.4 (Levine [6]). A topological space is $T_{\frac{1}{2}}$ if every g-closed set is closed.

REMARK 2.5. Levine proves in [6] that every T_1-space is $T_{\frac{1}{2}}$ and every $T_{\frac{1}{2}}$ space is T_0, although neither implication is reversible. In [4] Dunham establishes the following characterization:

THEOREM 2.6. $(X,\mathcal{T})$ is $T_{\frac{1}{2}}$ iff every singleton in X is either open or closed iff every subset of X is the intersection of all open sets and all closed sets containing it.

3. The generalized closure operator

DEFINITION 3.1. For a space $(X,\mathcal{T})$, let $\mathcal{D}=\{A : A\subseteq X \text{ and } A \text{ is } g\text{-closed}\}$.

DEFINITION 3.2. For any $E\subseteq X$, define $c^*(E)=\cap\{A : E\subseteq A\in\mathcal{D}\}$.

LEMMA 3.3. *If* $E\subseteq X$, *then* $E\subseteq c^*(E)\subseteq c(E)$.

PROOF. A closed set is g-closed.

REMARK 3.4. Both containment relations in the previous lemma may be proper. Consider $X=\{a,b,c\}$ with topology $\mathcal{T}=\{\phi, \{a\}, \{a,b\}, X\}$. Then $c^*(\{a\})=\{a,c\}$ since the only g-closed supersets of $\{a\}$ are $\{a,c\}$ and X, while $c(\{a\})=X$. That is, $\{a\}\subsetneqq c^*(\{a\})\subsetneqq c(\{a\})$.

THEOREM 3.5. c^* *is a Kuratowski closure operator on* X.

PROOF. (i) $c^*(\phi)=\phi$ and $E\subseteq c^*(E)$ follow from Lemma 3.3.

(ii) If $E_1\cup E_2\subseteq A\in\mathcal{D}$, then $E_i\subseteq A$ and so $c^*(E_i)\subseteq A$ for $i=1,2$. Thus $c^*(E_1)\cup c^*(E_2)\subseteq\cap\{A:E_1\cup E_2\subseteq A\in\mathcal{D}\}=c^*(E_1\cup E_2)$. Conversely, we assert that $c^*(E_1\cup E_2)\subseteq c^*(E_1)\cup c^*(E_2)$. For, if there is an $x\in c^*(E_1\cup E_2)$ with $x\notin c^*(E_1)\cup c^*(E_2)$, then there are g-closed sets A_1 and A_2 with $E_1\subseteq A_1$, $E_2\subseteq A_2$, and $x\notin A_1\cup A_2$. But then $E_1\cup E_2\subseteq A_1\cup A_2$, a g-closed set by Theorem 2.3, contradicting $x\in c^*(E_1\cup E_2)$. We conclude that $c^*(E_1\cup E_2)=c^*(E_1)\cup c^*(E_2)$.

(iii) Finally, if $E\subseteq A\in\mathcal{D}$, then $c^*(E)\subseteq A$ and $c^*(c^*(E))\subseteq A$ by definition of c^*. Hence $c^*(c^*(E))\subseteq\cap\{A : E\subseteq A\in\mathcal{D}\}=c^*(E)$.

By (i)-(iii), c^* is a closure operator on X.

DEFINITION 3.6. Let $\mathcal{T}^*$ be the topology on X generated by c^* in the usual manner. That is, $\mathcal{T}^*=\{O^* : c^*(\mathcal{C}O^*)=\mathcal{C}O^*\}$.

THEOREM 3.7. $\mathcal{T}\subseteq\mathcal{T}^*$ *with equality iff* $(X,\mathcal{T})$ *is* $T_{\frac{1}{2}}$.

PROOF. If E is $\mathcal{T}$-closed, $E\subseteq c^*(E)\subseteq c(E)$ implies that E is $\mathcal{T}^*$-closed. Thus $\mathcal{T}\subseteq\mathcal{T}^*$. Further, suppose $\mathcal{T}=\mathcal{T}^*$ and let $A\subseteq X$ be g-closed in $(X,\mathcal{T})$. Then $A=c^*(A)$ and so A is closed in $\mathcal{T}^*=\mathcal{T}$. Thus $(X,\mathcal{T})$ is $T_{\frac{1}{2}}$. Conversely, if $(X,\mathcal{T})$ is $T_{\frac{1}{2}}$, its closed sets and g-closed sets coincide and so $c=c^*$.

Hence $\mathscr{T} = \mathscr{T}^*$.

THEOREM 3.8. *For any space $(X, \mathscr{T})$, $x \neq y$ implies $c^*(x) \neq c^*(y)$.*

PROOF. If $\{x\}$ is closed, $y \notin \{x\} = c\{x\} = c^*(x)$. Otherwise, $y \in \mathscr{C}\{x\}$, a g-closed set by Theorem 2.2. Thus $y \in c^*(y) \subseteq \mathscr{C}\{x\}$, and so $x \notin c^*(y)$.

REMARK 3.9. The previous result shows that $(X, \mathscr{T}^*)$ is always T_0. In fact, we can establish a stronger result:

THEOREM 3.10. *For any space $(X, \mathscr{T})$, $(X, \mathscr{T}^*)$ is $T_{\frac{1}{2}}$.*

PROOF. If $\{x\}$ is $\mathscr{T}$-closed, $\{x\}$ is $\mathscr{T}^*$-closed as well. Otherwise, $\mathscr{C}\{x\}$ is g-closed and so $c^*(\mathscr{C}\{x\}) = \mathscr{C}\{x\}$, which implies that $\{x\}$ is $\mathscr{T}^*$-open. By Theorem 2.6, $(X, \mathscr{T}^*)$ is $T_{\frac{1}{2}}$.

REMARK 3.11. Based on the conclusion of Theorem 3.10, we shall designate $\mathscr{T}^*$ as the "$T_{\frac{1}{2}}$ extension" of $\mathscr{T}$. We deduce immediately:

COROLLARY 3.12. *For any topology, $(\mathscr{T}^*)^* = \mathscr{T}^*$.*

PROOF. $(X, \mathscr{T}^*)$ is $T_{\frac{1}{2}}$ and thus $(\mathscr{T}^*)^* = \mathscr{T}^*$ by Theorem 3.7.

4. Some properties of the $T_{\frac{1}{2}}$ extension

EXAMPLE 4.1. The $T_{\frac{1}{2}}$ extension process does not necessarily preserve nor reverse the inclusion of topologies. Consider $X = \{a, b\}$ with $\mathscr{T} = \{\phi, X\}$ and $\mathscr{U} = \{\phi, \{a\}, X\}$. Then $\mathscr{T}^*$ is discrete and $\mathscr{U}^* = \mathscr{U}$ (since $(X, \mathscr{U})$ is $T_{\frac{1}{2}}$), and thus $\mathscr{T} \subseteq \mathscr{U}$ while $\mathscr{U}^* \subseteq \mathscr{T}^*$. However, if we now let $\mathscr{V}$ be the discrete topology on X, then $\mathscr{V}^* = \mathscr{V}$ and so $\mathscr{U} \subseteq \mathscr{V}$ while $\mathscr{U}^* \subseteq \mathscr{V}^*$.

REMARK 4.2. The difficulty encountered in the previous example results from the fact that the $T_{\frac{1}{2}}$ extension of both the discrete and indiscrete topology is discrete. We characterize discreteness of $(X, \mathscr{T}^*)$ in:

THEOREM 4.3. *The following conditions are equivalent*:

(a) *$(X, \mathscr{T}^*)$ is discrete.*

(*b*) *For each $x \in X$, $\mathscr{C}\{x\}$ is g-closed in $(X, \mathscr{T})$.*

(*c*) *If $\{x\}$ is $\mathscr{T}$-closed, $\{x\}$ is $\mathscr{T}$-open.*

PROOF. (a) implies (b): If $(X, \mathscr{T}^*)$ is discrete then, for each x, $\mathscr{C}\{x\} = c^*(\mathscr{C}\{x\}) = \bigcap\{A : \mathscr{C}\{x\} \subseteq A \in \mathscr{D}\}$. It follows that $\mathscr{C}\{x\}$ is itself g-closed in $(X, \mathscr{T})$.

(b) implies (c): Suppose $\{x\}$ is $\mathcal{T}$-closed. Then $\mathcal{C}\{x\}$ is $\mathcal{T}$-open and so $c(\mathcal{C}\{x\})\subseteq\mathcal{C}\{x\}$ by assumption (b). Hence, $\{x\}$ is $\mathcal{T}$-open as well.

(c) implies (a): If $\{x\}$ is $\mathcal{T}$-closed, $\{x\}\in\mathcal{T}$ and thus $\{x\}\in\mathcal{T}^*$ by Theorem 3.7. If $\{x\}$ is not $\mathcal{T}$-closed, $\mathcal{C}\{x\}$ is g-closed by Theorem 2.2 and again $\{x\}\in\mathcal{T}^*$.

EXAMPLE 4.4. Let X be an uncountable set with $\mathcal{T}=\{\phi, X\}$. Then $(X,\mathcal{T})$ is compact, connected, and second axiom, while $(X,\mathcal{T}^*)$, being discrete by Theorem 4.3, shares none of these properties.

REMARK 4.5. The final results of this paper will show that, as we put stronger "regularity" conditions on $(X,\mathcal{T})$, we induce stronger "separation" properties on $(X,\mathcal{T}^*)$. We first recall two definitions:

DEFINITION 4.6 (Davis [2]). $(X,\mathcal{T})$ is an *R_0-space* if $x\in O\in\mathcal{T}$ implies $c(x)\subseteq O$ (i.e., singletons are g-closed).

DEFINITION 4.7 (see Dunham [3]). $(X,\mathcal{T})$ is *weakly Hausdorff* if $c(x)=c(y)$ whenever there is a net $S: D\to X$ with $lim\ S=x$ and $lim\ S=y$.

REMARK 4.8. It is proved in [3] that any regular space is weakly Hausdorff and any weakly Hausdorff space is R_0, although neither implication is reversible.

THEOREM 4.9. *If $(X,\mathcal{T})$ is R_0, then $(X,\mathcal{T}^*)$ is T_1.*

PROOF. If $x\in X$, $\{x\}$ is g-closed and so $c^*(x)=\{x\}$.

THEOREM 4.10. *If $(X,\mathcal{T})$ is weakly Hausdorff, then $(X,\mathcal{T}^*)$ is T_2.*

PROOF. Let $S: D\to X$ be a net such that $limS=x$ and $limS=y$ in $(X,\mathcal{T}^*)$. We assert that $x=y$. For, by Theorem 3.7, $limS=x$ and $limS=y$ in $(X,\mathcal{T})$, a weakly Hausdorff space, and thus $c(x)=c(y)$. So, if either $\{x\}$ or $\{y\}$ is $\mathcal{T}$-closed, then $x=y$. If neither $\{x\}$ nor $\{y\}$ is $\mathcal{T}$-closed, then both $\mathcal{C}\{x\}$ and $\mathcal{C}\{y\}$ are g-closed by Theorem 2.2 and thus $\{x\}\in\mathcal{T}^*$ and $\{y\}\in\mathcal{T}^*$. But then the net S is eventually in $\{x\}\cap\{y\}\in\mathcal{T}^*$ and so $x=y$. It follows that $(X,\mathcal{T}^*)$ is T_2.

THEOREM 4.11. *If $(X,\mathcal{T})$ is regular, then $(X,\mathcal{T}^*)$ is T_3 (regular and T_1).*

PROOF. Since $(X,\mathcal{T}^*)$ is $T_{\frac{1}{2}}$, it suffices to prove regularity. Let $x\notin F^*$, where F^* is $\mathcal{T}^*$-closed.

CASE 1. Suppose $\{x\}$ is $\mathcal{T}$-closed. Since $x\notin F^*=c^*(F^*)$, $x\notin A$ for some $A\supseteq F^*$ with A g-closed in $(X,\mathcal{T})$. But then $A\subseteq\mathcal{C}\{x\}\in\mathcal{T}$ and so $c(A)\subseteq\mathcal{C}\{x\}$. Thus

$x \notin c(A)$ in the regular space $(X, \mathscr{T})$, and so there are disjoint open sets O_1 and O_2 in $\mathscr{T}$ with $x \in O_1$ and $c(A) \subseteq O_2$. Hence $x \in O_1 \in \mathscr{T}^*$ and $F^* \subseteq A \subseteq c(A) \subseteq O_2 \in \mathscr{T}^*$ with $O_1 \cap O_2 = \phi$.

CASE 2. Suppose now that $\{x\}$ is not $\mathscr{T}$-closed. By Theorem 2.2. $\mathscr{C}\{x\}$ is g-closed and thus $\{x\} \in \mathscr{T}^*$. We assert that $F^* \subseteq \mathscr{C}c^*(x)$ and thus choose $y \in F^*$ arbitrary. If $\{y\}$ is $\mathscr{T}$-closed, $x \notin \{y\}$ implies, by regularity of $(X, \mathscr{T})$, that there exist O_1 and O_2 with $x \in O_1 \in \mathscr{T}$ and $\{y\} \subseteq O_2 \in \mathscr{T} \subseteq \mathscr{T}^*$ with $O_1 \cap O_2 = \phi$, and so $y \in \mathscr{C}c^*(x)$; otherwise, if $\{y\}$ is not $\mathscr{T}$-closed, we have, by Theorem 2.2, $y \in \{y\} \in \mathscr{T}^*$ with $\{y\} \cap \{x\} = \phi$, and again $y \in \mathscr{C}c^*(x)$. This establishes the assertion that $F^* \subseteq \mathscr{C}c^*(x)$. But then $x \in \{x\} \in \mathscr{T}^*$ and $F^* \subseteq \mathscr{C}c^*(x) \in \mathscr{T}^*$ with $\{x\} \cap \mathscr{C}c^*(x) = \phi$.

By cases (1) and (2), $(X, \mathscr{T}^*)$ is regular and the theorem is proved.

REMARK 4.12. We summarize the results concerning the transfer of properties from $(X, \mathscr{T})$ to $(X, \mathscr{T}^*)$ in the following diagram:

$$\begin{array}{llllllll}
(X, \mathscr{T}): & \text{regular} & \longrightarrow & \text{weakly Hausdorff} & \longrightarrow & R_0 & \longrightarrow & \text{arbitrary} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(X, \mathscr{T}^*): & T_3 & \longrightarrow & T_2 & \longrightarrow & T_1 & \longrightarrow & T_{\frac{1}{2}}
\end{array}$$

EXAMPLE 4.13. Converses of the three previous theorems fail. For if $X = \{a, b, c\}$ with $\mathscr{T} = \{\phi, \{a\}, X\}$, then $(X, \mathscr{T}^*)$ is discrete by Theorem 4.3, while $(X, \mathscr{T})$ is not even R_0 since $\{a\}$ is not g-closed.

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