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GENERALIZED UNITS: THE POINT-LIKE SETS

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It was proved by M.H. Stone [4] that by identifying "undistinguishable" points in a topological space every space can be made into a *To*-space. C.E. Aull and W.J. Thron [1] showed that every topological space X is partitioned into the sets of the form $\langle x \rangle$, for each $x \in X$; where each $\langle x \rangle$ is intersection of $\overline{\{x\}}$ (the closure of $\{x\}$) and the Kernel of x ($\equiv$the set of all y for which x is not weakly separated from y). And they proved the following theorem:

In a topological space (X,T), if R is the equivalence relation on $X\times X$ defined by $(x, y)\in R$ iff $y\in\langle x\rangle$, then X/R with its quotient topology T_R is a T_0-space, and the spaces (X, T) and $(X/R, T_R)$ are lattice equivalent.

A.S. Davis [2] suggested a classification scheme for separation axioms R_0, R_1 and Regularity, which is offered as a natural extension of a remark made by J.L. Kelley [3] (page 130) on pseudo matrics. In R_0 and R_1 spaces, $\langle x\rangle = \overline{\{x\}}$, for each x in the space. R_0 and R_1 spaces are lattice equivalent to their respective quotient spaces, obtained by equivalence relation R defined in the above mentioned theorem.

All these indicate the point-like behaviour of sets $\langle x\rangle$, for x in a topological space X. In this paper we study the properties of these sets $\langle x\rangle$ and show that they behave like points.

1. Definition and elementary properties

DEFINITION. A non void subset U in a topological space X is said to be a *pseudo generalized unit* (written as $p.q.u.$) if for each open set O in X; either $U\subset O$ or $U\cap O=\phi$.

DEFINITION. A maximal pseudo generalized unit in a topological space X is said to be a *generalized unit* (written as g. unit).

DEFINITION. A closed pseudo generalized unit in a topological space X is said to be a *unit*.

REMARKS. (a) Define the relation '$\cong$' in a topological space X as follows:

For $x, y \in X$, $x \cong y$ provided that for each open set O in X, $x \in O$ if and only if $y \in O$. Then '$\cong$' is an equivalence relation on X and each equivalence class is precisely a $g.$ unit in X. Then clearly for x in X, the set $\langle x \rangle$ is a $g.$ unit.

(b) If u is a $g.$ unit in a space X, then following is obvious: for each pair $x, y \in u$; (i) x is interior, frontier, exterior point of a subset A of X if and only if y is so respectively; (ii) x is a limit point of A if and only if y is a limit point of A, provided that $A \cap u \neq \phi$ implies $A \cap u$ is not a singleton; (iii) x is a condensation point if and only if y is so.

(c) There is a unique $g.$ unit containing a given pseudo $g.$ unit.

(d) If u is a unit; complement of $u (\equiv \varepsilon(u))$ is open. And if u is a proper subset of $u' \subset X$; then neither $u' \subset \varepsilon(u)$ nor $u' \cap \varepsilon(u) = \phi$. This shows that u is a maximal $p.g.u.$. Hence it follows that every unit is a $g.$ unit; but the following example shows that a $g.$ unit may not be a unit.

Let $X = \{a, b, c, d, e\}$ and

$$T = \{X, \phi, \{a, b, c\}, \{d, e\}, \{a, b\}, \{a, b, d, e\}\}.$$

Then $\{a, b\}$ is a $g.$ unit in X; but it is not a unit.

(e) For each closed set F in X, if u is a $g.$ unit, then either $u \subset F$ or $u \cap F = \phi$. From this it follows that if u_α and u_β are distinct $g.$ units then either $\bar{u}_\alpha \supset u_\beta$ or $\bar{u}_\alpha \cap u_\beta = \phi$.

(f) A non singleton $g.$ unit is dense in itself and if it is a unit, it is perfect.

(g) $g.$ units are compact, connected, path-wise connected etc.

(h) The intersection of all open sets O_i, containing a $g.$ unit u, may contain other $g.$ units; but u can be the only possible unit (if it is a unit) contained in the intersection.

THEOREM 1. *The total intersection of every maximal nest in the family of closures of all singletons under order defined by inclusion, in a topological space X, is a unit, if it is not empty.*

PROOF. Consider the family $\mathscr{A} = \{\overline{\{x\}}\}_{x \in X}$. By Hausdorff maximal principle, for each $x \in X$ there is a maximal nest ${}_{\delta}\gamma_x = \{\overline{\{x_i\}}\}$, $i \in I$ in $\mathscr{A}$, which includes $\overline{\{x\}}$.

Then to show that $\bigcap_i \overline{\{x_i\}}$ is either empty or a unit. Suppose that $\bigcap_i \overline{\{x_i\}} (\neq \phi)$ is not a unit, since $\bigcap \overline{\{x_i\}}$ is closed there should be an open set O in X such that

(i) $O\cap\{\bigcap_i \overline{\{x_i\}}\}\neq\phi$ and also (ii) $(x-O)\cap\{\bigcap_i \overline{\{x_i\}}\}\neq\phi$.

Thus $\{X-O\}\cap\{\bigcap_i \overline{\{x_i\}}\}$ is closed and not empty.

Take $y\in\{X-O\}\cap\{\bigcap_i \overline{\{x_i\}}\}$.

Also because of (i), $\{X-O\}\cap\{\bigcap_i \overline{\{x_i\}}\}$ is contained in $\bigcap_i \overline{\{x_i\}}$ properly, therefore $\overline{\{y\}}\subset\bigcap_i \overline{\{x_i\}}$ properly.

Clearly $\overline{\{y\}}\notin{}_{\delta}\gamma_x$, because otherwise $\bigcap_i \overline{\{x_i\}}$ can not contain $\overline{\{y\}}$ properly.

Thus ${}_{\delta}\gamma_x\cup\{\overline{\{y\}}\}$, which is obviously a nest in $\mathscr{A}$, contains ${}_{\delta}\gamma_x$ properly. But this contradicts the maximality of ${}_{\delta}\gamma_x$ in $\mathscr{A}$. And hence our assumption that $\bigcap_i \overline{\{x_i\}}$ is not a unit, is not true.

2. Generalized units of subspaces and product spaces

THEOREM 2. *If u is a g. unit in a top. space X, then $u\cap Y$ (if it is not void) is a g. unit in a subspace Y of X. Also if u' is a g. unit in a subspace Y of X; then there is a g. unit u in X such that $u\cap Y=u'$.*

PROOF. u is a g. unit in a space $X\Rightarrow$ for each open set O in X, either $u\subset O$ or $u\cap O=\phi\Rightarrow$ for each open set $O\cap Y$ in Y, either $u\cap y\subset O\cap y$ or $\{u\cap Y\}\cap\{O\cap Y\}=\phi$. Now if possible let u' be the maximal $p.\ g.$ unit in Y such that $u'\supset u\cap Y$ properly. Then $u\cup u'$ contains u properly and for each open set O in X either $(u\cup u')\subset O$ or $(u\cup u')\cap O=\phi$; because $u\subset O$ if and only if $u'\subset O$ and $u\cap O=\phi$ if and only if $u'\cap O=\phi$. Thus it contradicts the maximality of u in X i.e. $u\cap Y$ is maximal in Y. Thus $u\cap Y$ is a g. unit in Y.

Now, let u' be a g. unit in a subspace Y of X. Let $y\in u'$. Let u be the g. unit in X, which contains y. Then $u\cap Y$ is a g. unit in Y as proved above. But since $y\in u'$ and $y\in u\cap Y$, it implies that $u\cap Y=u'$; because there is a unique g. unit in Y containing y.

LEMMA 1. *u is a g. unit in X iff u is a maximal set satisfying the property that for all basic open sets G of X; either $u\subset G$ or $u\cap G=\phi$.*

PROOF. Follows immediately.

THEOREM 3. *In a non-empty product space $\prod_i X_i$, $\prod_i u_i$ is a g. unit iff u_i is a g. unit in the topological space X_i, for each $i\in I$.*

PROOF. Let u_i be a g. unit in X_i for each $i\in I$. Let $u=\prod_i u_i$. Take any basic open set $G=\prod_i O_i$ of $X=\prod_i X_i$ [Here only finite number of O_i's are open sets in X_i other than X_i].

For any i, since O_i is an open set;
either $u_i\subset O_i$ or $u_i\cap O_i=\phi$.

Therefore, either (i) $u_i\subset O_i$ for each i or

(ii) there is an i_0 such that $O_{i_0}\cap u_{i_0}=\phi$.

In case (i) $u\subset G$. In case (ii) $u\cap G=\phi$. Thus for basic open set G in X, either $u\subset G$ or $u\cap G=\phi$, i.e. u is a $p.g.$ unit in X.

Now, if possible let u' be another $p.g.$ unit in X containing u properly. Let projection of u' on X_i be denoted as $[u']_i$. As u_i is the projection of u on X_i, there is a $j\in I$ such that $[u']_j\supset u_j$ properly. Now since u_j is a g. unit and since $[u']_j\supset u_j$ properly, there is an open set O_j in X_j such that neither $O_j\cap [u']_j=\phi$ nor O_j contains $[u']_j$. Then the basic open set $z(O_j)=\{x\mid x\in X$ and $x_j \in O_j\}$ intersects u' and does not contain u'. And therefore u' can not be a $p.g.$ unit in X. Thus u is a maximal $p.g.$ unit and hence a g. unit in X.

To prove the converse, let $\prod_i u_i$ be a g. unit in $\prod_i X_i$. Take any $j\in I$. Consider any open set O_j in X_j. Consider $z(O_j)=\{x\mid x\in X$ and $x_j\in O_j\}\}$ a basic open set in X. As $\prod_i u_i$ is a g unit in X,

either (i) $\prod_i u_i\subset z(O_j)$

or (ii) $(\prod_i u_i)\cap z(O_j)=\phi$.

In case (i) $u_j\subset O_j$ and in case (ii) $u_j\cap O_j=\phi$. Hence u_j is a $p.$ $g.$ unit in X_j.

This is true for each $j\in I$.

To prove maximality of u_j, let u'_j contain u_j properly and let u'_j be a $p.$ $g.$ unit in X_j. Then $\prod_i V_i$; where $V_i=u_i$ for $i\neq j$ and $V_j=u'_j$, contains $\prod_i u_i$ properly. Now each $V_i(i\neq j)$ is a $p.g.$ unit by (II) and $V_j=u'_j$ is a $p.$ $g.$ unit by our hypothesis. Then, as we have proved in (I), we can prove that $\prod_i V_i$ is a $p.g.$ unit in $\prod_i X_i$. But since $\prod_i u_i$ is a g. unit in $\prod_i X_i$, this contradicts the maximality of $\prod_i u_i$. Therefore u_j is a maximal $p.$ $g.$ unit and hence it is a g. unit in X_j, for each $j\in I$.

3. Generalized units and convergence

We shall now introduce the concept of convergence of a sequence of g. units in a topological sapce. The following Lemma (proof of which is simple) is useful in introducing the concept.

LEMMA 2. *Let (u_n) be a sequence of g. unit in a topological space X. Let (x_n) and (y_n) be sequences of points such that x_n, $y_n \in u_n$ for each n. Then lim $(x_n) = lim(y_n)$. Also if $x \in lim(x_n)$, the g. unit containing x is contained in lim (x_n).*

A similar result can be proved for a 'net' of *g.* units. These two results make the following two definitions meaningful.

DEFINITION. A sequence (u_n) of *g.* units in a topological space *X* is said to *converge to* a *g.* unit u_x in *X*, and written as $u_n \longrightarrow u_x$; if for each point $x_n \in u_n$, the sequence (x_n) converges to each point $x \in u_x$.

DEFINITION. A net (u_λ) of *g.* units u_λ in a topological space *X* is said to *converge to* a *g.* unit u_x in *X*, and written as $u_\lambda \longrightarrow u_x$; if for each point $x_\lambda \in u_\lambda$, the net (x_λ) converges to each point $x \in u_x$.

It can be similarly shown that if a point *x* lies in the limit of a filter $\mathscr{F}$ on a topological space *X*, then the *g.* unit *u* containing *x* will be in lime $\mathscr{F}$. We may therefore give the following definition.

DEFINITION. A filter $\mathscr{F}$ on a topological space *X* is said *to converge to* a *g.* unit u_x in *X*, if for all the neighbourhoods of some $x \in u_x$ are members of $\mathscr{F}$.

4. Generalized units and subsets

DEFINITION. A non-empty subset *A* of a topological space *X* is said to be a *u-completion in X*, if it is a union of *g.* units of *X*; and *A* is *u*-incompletion in *X* otherwise i. e. *A* is *u*-incompletion in *X* provided that there is a *g.* unit *u* in *X* such that $A \cap u \neq \phi$ and $(X-A) \cap u \neq \phi$.

DEFINITION. If $A \subset X$ is a *u*-incompletion in a topological space *X*, the set $\bigcup_\alpha \{u_\alpha \mid A \cap u_\alpha \neq \phi,\ u_\alpha$ is a *g.* unit in $X\}$ is said to be the *u-completion* of *A* in *X*, and is denoted by A^*.

DEFINITION. Let *A* be any subset in a topological space *X*. Then the set $\bigcup_\alpha \{u_\alpha \mid A \cap u_\alpha \neq \phi,\ (X-A) \cap u_\alpha \neq \phi$ and u_α is a *g.* unit in $X\}$ is said to be the *Boarder* of *A* and is denoted by $Bdr(A)$ or $Bdr\ A$.

Then the following results are easy to prove.

(a) *For each non-empty subset A in a space X, BdrA and A-BdrA are u-completions and they are disjoint. Also for* $A\neq\phi\neq B$, $[A\cap B]^*\cup[BdrA\cap BdrB]=A^*\cap B^*$.

(b) $[\bigcup_\alpha A_\alpha]^*=\bigcup_\alpha A_\alpha{}^*$.

(c) $(A-B)^*-BdrB=A^*-B^*$.

(d) *In a topological space X all open sets and all closed sets are u-completions.*

(e) *All* G_δ *sets and all* F_δ *sets are u-completions.*

(f) *The set of all u-completions is a σ-ring and hence every Borel set is a u-completion.*

THEOREM 4. *Let A be any u-incompletion in a topological space X. Let* A^* *be its u-completion in X. Let* $x\in X\text{-}Bdr(A)$. *Then*

(i) *x is a limit point of A if and only if x is a limit point of* A^*.

(ii) *If* $x\in X-\overline{Bdr(A)}$, *x is an interior point of A if and only if x is an interior point of* A^*.

(iii) *x is an exterior point of A if and only if x is an exterior point of* A^*.

(iv) *x is an isolated point of A if and only if x is an isolated point of* A^*.

PROOF. (i) '$\Longrightarrow$' part is obvious.

We prove '$\Longleftarrow$' part.

x is a limit point of $A^*\Longrightarrow$ for each open set O containing x, $O\cap(A^*-\{x\})\neq\phi\Longrightarrow O\cap(A^*-\{x\})\neq\phi\,[x\notin Bdr(A)]$.

i.e. x is a limit point of A.

(ii) '$\Longrightarrow$' part is obvious.

To prove '$\Longleftarrow$' part.

$x\notin\overline{Bdr(A)}$ and x is an interior point of $A^*\Longrightarrow$ there is an open set O containing x such that $O\subset A^*$ and $x\in X-\overline{Bdr(A)}=G$, which is open$\Longrightarrow x\in G\cap O$ and $G\cap O\subset A$: because $G\cap Bdr(A)=\phi$ and $O\subset A^*=A\cup Bdr(A)$.

i.e. x is an interior point of A.

(iii) '$\Longleftarrow$' part is clear.

To prove '$\Longrightarrow$' part, see that if the open set O containing x, is contained in $X-A$; then it is also contained in $X-A^*$.

(iv) For an open set O_x containing x, and $x\notin Bdr(A)$,

$$O_x\cap(A^*-\{x\})\neq\phi\Longrightarrow O_x\cap(A-\{x\})\neq\phi$$

and $$O_x\cap(A^*-\{x\})=\phi\Longrightarrow O_x\cap(A-\{x\})=\phi.$$

Now, the proof is obvious.

THEOREM 5. *If A^* is the u-completion of a u-incompletion A in a topological space X,*

(i) $\overline{A}=\overline{A^*}$ *and hence A is dense iff A^* is dense*

(ii) *A is no-where dense iff A^* is no-where dense.*

(iii) *For any subset B of X,*

$$\overline{A}\cap B=\phi=A\cap\overline{B} \text{ if and only if}$$
$$\overline{A^*}\cap B=\phi=A^*\cap\overline{B}.$$

(iv) *A is of first category if and only if A^* is of first category.*

PROOF. (i) For each open set O containing x and $x\notin A$

$$O\cap(A^*-\{x\})\neq\phi\Longrightarrow O\cap(A-\{x\})\neq\phi$$

i.e. if $x\notin A$, and $x\in\overline{A^*}\Longrightarrow x\in\overline{A}$ and $x\in A\Longrightarrow x\in\overline{A}$ always, these together show that $\overline{A^*}\subset\overline{A}$. Also since $A\subset A^*$, $\overline{A}\subset\overline{A^*}$ i.e. $\overline{A}=\overline{A^*}$. The rest follows immediately.

(ii) Consequence of (i).

(iii) Since $\overline{B}$ is closed, for each g. unit u; either $u\subset\overline{B}$ or $u\cap\overline{B}=\phi$ and since $\overline{A}=\overline{A^*}$, the proof is easily followed.

(iv) The definition of first category set and the result (ii) imply the statement easily.

We recall that two topological spaces (X,T) and (Y,u) are lattice equivalent if and only if a one-to-one and onto order preserving map can be established between the elements of T and those of u.

THEOREM 6. *If A^* is the u-completion of a subset A in a topological space X, then A and A^* are lattice equivalent as subspaces of X.*

PROOF. Let G be open in X. Let u_α be g. units in X. First see that

(i) $G\cap A=\phi$ if and only if $G\cap A^*=\phi$

and (ii) $G\cap A$ is open in A if and only if $G\cap A^*$ is open in A^*.

Then the proof is obvious.

5. Miscellaneous results and concluding remark

THEOREM 7. *Let $\{T_\alpha\}_{\alpha\in I}$ be a collection of topologies on a set X. Let T be a topology in X, generated by $\bigcup_\alpha T_\alpha$ as subbasis. Let u be any g. unit in (X,T). Let $x\in u$. Let u_α be any g. unit in (X,T_α) for each α, and let $x\in u_\alpha$ for all α. Then $u=\bigcap_\alpha u_\alpha$.*

PROOF. Clearly for each $O\in T$, either $O\supset u$ or $O\cap u=\phi\Longrightarrow$ for each $O\in\bigcup_\alpha T_\alpha$, either $O\supset u$ or $O\cap u=\phi \Longrightarrow u$ is a pseudo $g.$ unit in each $(X, T_\alpha) \Longrightarrow u\subset \bigcap_\alpha u_\alpha$, because $x\in u$ and $x\in u_\alpha$ for all α. Now $x\in u_\alpha$ for all $\alpha \Longrightarrow \bigcap_\alpha u_\alpha \neq \phi$. Then for each α and each $O\in T_\alpha$; either $u_\alpha \subset O$ or $u_\alpha \cap O=\phi \Longrightarrow$ for each α and for each $O\in T_\alpha$, either $\bigcap_\alpha u_\alpha \subset O$ or $\{\bigcap_\alpha u_\alpha\}\cap O=\phi \Longrightarrow$ for each $O\in \bigcup_\alpha T_\alpha$, either $\bigcap_\alpha u_\alpha \subset O$ or $(\bigcap_\alpha u_\alpha)$ $\cap O=\phi$ and since O is a subbasic open set, this implies that for each $O\in T$, either $\bigcap_\alpha u_\alpha \subset O$ or $\{\bigcap_\alpha u_\alpha\}\cap O=\phi$ i.e. $\bigcap_\alpha u_\alpha$ is a pseudo $g.$ unit in $(X, T) \Longrightarrow \bigcap_\alpha u_\alpha$ $\subset u$, because $x\in \bigcap_\alpha u_\alpha$ and $x\in u$. Thus $\bigcap_\alpha u_\alpha = u$.

THEOREM 8. *Let (X, T_α) and (X, T_β) be topological spaces. Let $T_\alpha \subset T_\beta$. If u_α and u_β are g. units in (X, T_α) and (X, T_β) respectively, then*

$$u_\alpha = \bigcup_{x\in u_\alpha} u_\beta(x) \textit{ such that } x\in u_\beta(x).$$

PROOF. Since $T_\alpha \subset T_\beta$, each u_β is a $p.g.$ unit in (X, T_α). Then clearly,

$$u_\alpha = \bigcup_{x\in u_\alpha} u_\beta(x) \text{ such that } x\in u_\beta(x).$$

In the forthcoming papers, which shall appear elsewhere, the authors have defined and studied unit-wise separation pestulates and unit-wise topological equivalence of two topological spaces.

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