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ON BITOPOLOGICAL C-COMPACTNESS

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Introduction

It is well known that every continuous map from a compact space to a Hausdorff space is closed. Viglino [8], initiated the study of a larger class of spaces, called *C-compact spaces*, for which this property still holds. The idea of C-compactness was generalized independently to bitopological spaces by the author ([5], Chapter 3.2 and [6]) and by Vasudevan and Goel [7]. The definitions are different and both lead to notions more general than Fletcher, Hoyle and Patty's *pairwise compactness* [3]. In common with the latter they are not product invariant, have the "maximal compact" and "minimal Hausdorff" property and each may be characterized in terms of the adherent convergence of certain open filter bases. In [2], Cooke and Reilly compared various notions of bitopological compactness and in [6] the author gave a bitopological compactness implications diagram based on the results in [2] and [6] and including his notion of *pairwise C-compactness*. The diagram is extended below to include Vasudevan and Goel's notion of pairwise C-compactness. Terminology is as in [2].

DEFINITION. Let (X, T_1, T_2) be a bitopological space, $A \neq X$ any T_i-closed subset of X and U any T_j-open cover of A.

(1) If it is always possible to find a finite subfamily $\{U_1, \cdots, U_n\}$ of U such that

$$A \subset \mathrm{cl}_{T_i}\left(\bigcup_{k=1}^{n} U_k\right)$$

we say X is *pairwise C-compact* ([5], [6]).

(2) If it is always possible to find a finite subfamily $\{U_1, \cdots, U_n\}$ of U such that

$$A \subset \mathrm{cl}_{T_j}\left(\bigcup_{k=1}^{n} U_k\right)$$

we say X is *VG-pairwise C-compact* ([7]).
In each case i, $j \in \{1, 2\}$ and $i \neq j$.

The following is an example of a pairwise C-compact space which is not VG-

pairwise C-compact.

EXAMPLE 1. Let $X=[0,1]$, T_1=usual topology on X, and $T_2=\{\phi, X, \{1\}\}$. To show that (X, T_1, T_2) is pairwise C-compact, consider the T_2-closed set $[0,1)$ and any T_1-open cover U of it. If U does not contain X then one of its members must contain a set of the form $[0,b)$ for some $0<b\leq 1$ and then $\mathrm{cl}_{T_2}[0,b)=[0,1)$. It is now easy to see that X is pairwise C-compact. On the other hand, the T_2-closed set $[0,1)$ has T_1-open cover $U=\{[0,b)\,|\,b\in(0,1)\}$ and the T_1-closure of the union over any finite subfamily of U is $[0,b]$, some $0<b<1$, which does not contain $[0,1)$.

The next example gives a VG-pairwise C-compact space which is not pairwise C-compact.

EXAMPLE 2. [4]. Let $X=[-1,0)\cup(0,1]$ and consider (X, L, R) where L and R are the left ray and right ray topologies induced on X. Consider the L-closed set $(0,1]$ with R-open cover $U=\left\{\left(\frac{1}{n}, 1\right]\middle|\, n\in N,\ n\geq 2\right\}$. The L-closure of any finite union of members of U is $\left[\frac{1}{m}, 1\right]$, some positive integer m, which does not contain $(0, 1]$. Hence (X, L, R) is not pairwise C-compact. However, (X, L, R) is VG-pairwise C-compact: consider any proper L-closed set A. Notice that $1\in A$ so that some member of any R-open cover of A must contain 1 and $\mathrm{cl}_R\{1\}=X$. The result follows by symmetry.

The final example gives a space which is *B-compact*, i.e. every T_i-open cover of the space has a finite T_j-open subcover, $i, j\in\{1, 2\}$ and $i\neq j$ ([1], [2]), but is not pairwise C-compact in either sense.

EXAMPLE 3. Let $X=[0, 1]$.

$$T_1=\{\phi, X, \{0\}\}\cup\{[0, a)\,|\,0<a\leq 1\}$$
$$T_2=\{\phi, X, \{1\}\}\cup\{(a, 1]\,|\,0\leq a<1\}$$

Then (X, T_1, T_2) is B-compact ([2], Example 3) but is not pairwise C-compact or VG-pairwise C-compact: consider the T_1-closed set $A=(0, 1]$ and the T_2-open cover $U=\{(a, 1]\,|\,0<a<1\}$ of A. Any union over a finite subfamily of U is of the form $(a, 1]$, some $0<a<1$ and neither the T_1-closure nor the T_2-closure of $(a, 1]$ contains A.

In view of the results in [2], [6], [7] and the above examples the following implications diagram holds where no arrows can be reversed and no others fitted.

$$
\begin{array}{ccc}
 & & VG\text{-pairwise } C\text{-compact} \\
 & & \uparrow \\
\text{finite} \rightarrow & \text{semi-compact} \rightarrow & \text{pairwise compact} \\
 & \downarrow & \downarrow \\
 & \text{pseudo-compact} & \text{pairwise } C\text{-compact} \\
 & \uparrow & \\
 & B\text{-compact} &
\end{array}
$$

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