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THE T_θ-TOPOLOGY AND FAINTLY CONTINUOUS FUNCTIONS

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1. Introduction

For a topological space X and $A \subset X$, the *θ-closure* of A is defined [9] to be the set of all $x \in X$ such that every closed neighborhood of x intersects A nonemptily and is denoted by $Cl_\theta(A)$. The subset A is called *θ-closed* if $Cl_\theta(A)=A$. In a similar manner, the *θ-interior* of a set $A \subset X$ is defined to be the set of all $x \in A$ for which there exists a closed neighborhood of x contained in A. The θ-interior of A is denoted by $\mathrm{Int}_\theta(A)$. In particular, the concept of θ-closed sets has been extensively studied by Professors Velicko [9], Dickman and Porter [1], Joseph [3] and others. With the definition of the θ-interior of a set, a new topology will be described which is related to the semi-regular topology on (X, T). The semi-regular topology, denoted by T_s, is the topology having as its base the set of all regular-open sets in (X, T) [2, Problem 22, p. 92]. Recall that a set A is regular-open provided $\mathrm{Int}(\mathrm{Cl}(A))=A$. Specifically, for any set A, $\mathrm{Int}(\mathrm{Cl}(A))$ is always regular-open.

2. The T_θ-topology

DEFINITION 1. An open set U in (X, T) is called *θ-open* if $\mathrm{Int}_\theta(U)=U$.

From the definition of θ-closed sets, it follows that the complement of a θ-open set is θ-closed and the complement of a θ-closed set is θ-open. According to [9], the intersection of θ-closed sets is θ-closed and the finite union of θ-closed sets is a θ-closed set. Therefore, arbitrary unions and finite intersections of θ-open sets are themselves θ-open. Consequently, the collection of θ-open sets in a topological space (X, T) form a topology T_θ on X which we call the *T_θ-topology*. Evidently, $T=T_\theta$ if and only if (X, T) is regular.

THEOREM 1. *Let X be any topological space. If $V \subset X$ is θ-open and $x \in V$, then there exists a regular-open set U such that $x \in U \subset Cl(U) \subset V$.*

PROOF. Since V is θ-open, there exists an open set W such that $x \in W \subset \mathrm{Cl}(W) \subset V$. But $\mathrm{Int}(\mathrm{Cl}(W))=U$ is regular-open and it follows that $x \in U \subset \mathrm{Cl}(U) \subset V$ due to the fact that $\mathrm{Cl}(W) \subset V$.

COROLLARY TO THEOREM 1. *The set V is θ-open if and only if for each $x\in V$ there exists a regular-open U such that $x\in U\subset Cl(U)\subset V$.*

Theorem 1 implies that in any topological space, $T_\theta\subset T_s$. The converse need not be true as the next example shows.

EXAMPLE 1. The topologies T_s and T_θ may be different even in a completely Hausdorff space. Let $X=(0,2)$ be a subset of the reals R with the usual topology. For each $k\in N$, define $H_k=\bigcup\left\{\left(\frac{2n+1}{2n(n+1)}, \frac{2n-1}{2n(n-1)}\right): n>k, \ n \text{ even}\right\}$ and topologize X using the following subbasic open sets: $\{V\subset X-\{1\} : V \text{ open in } R\}\cup\{H_k\cup G : k\in N, \ G\subset X, \ G \text{ open in } R \text{ and contains the point } 1\}$. Then $U=(3/4, 3/2)\cup H_1$ is regular-open, but not θ-open. Consequently, $T_s\neq T_\theta$.

THEOREM 2. *Let $A\subset X$ be θ-closed and let $x\in A$. Then there exists regular-open sets which separate x and A.*

PROOF. Since $X-A$ is θ-open and contains x, there exists a regular-open set U such that $x\in U\subset \mathrm{Cl}(U)\subset V$ by Theorem 1. Now $\mathrm{Int}(\mathrm{Cl}(X\text{-}\mathrm{Cl}(U)))$ is nonempty, regular-open, contains A and is disjoint from U.

A space is defined to be *almost-regular* [8] if for each $x\in X$ and regular-closed A not containing x, there exist disjoint open sets U and V such that $x\in U$ and $A\subset V$.

THEOREM 3. *Let X be almost-regular. Then each regular-open set in X is also θ-open.*

PROOF. Since X is almost-regular, for each regular-open V in X and $x\in V$ there exists a regular-open U such that $x\in U\subset \mathrm{Cl}(U)\subset V$ according to Theorem 2.2 of [8]. Thus each point of V has a closed neighborhood contained in V implying that V is θ-open.

COROLLARY TO THEOREM 3. *If (X,T) is almost-regular, then $T_s=T_\theta$.*

PROOF. By Theorem 3, $T_s\subset T_\theta$ and by Theorem 1, $T_\theta\subset T_s$. Therefore, $T_s=T_\theta$.

THEOREM 4. *The space (X,T) is almost-regular if and only if $T_s=T_\theta$.*

PROOF. If (X,T) is almost-regular, then $T_s=T_\theta$ by the Corollary to Theorem 3. Conversely, if $T_s=T_\theta$, let V be a regular-open set in (X,T) and let $x\in V$. Then V is also θ-open and by Theorem 1 there exists a regular-open set U such that $x\in U\subset \mathrm{Cl}(U)\subset V$. Consequently, (X,T) is almost-regular by Theorem 2.2

of [8].

THEOREM 5. *Let X and Y be topological spaces. If $U \subset X$ and $V \subset Y$ are θ-open, then $U \times V$ is θ-open in $X \times Y$.*

PROOF. Let $(x, y) \in U \times V$. Then there exist open sets U_1 and V_1 such that $x \in U_1 \subset \mathrm{Cl}(U_1) \subset U$ and $y \in V_1 \subset \mathrm{Cl}(V_1) \subset V$ because both U and V are θ-open. Therefore, $(x, y) \in \mathrm{Cl}(U_1) \times \mathrm{Cl}(V_1) = \mathrm{Cl}(U_1 \times V_1) \subset U \times V$. Consequently, each point of $U \times V$ has a closed neighborhood contained in $U \times V$ which shows $U \times V$ is θ-open.

THEOREM 6. *Let W be θ-open in the product space $\prod_{\alpha \in \Delta} X_\alpha$. Then each projection $\Pi_\alpha(W)$ is θ-open in X_α.*

PROOF. Let $y_\alpha \in \Pi_\alpha(W)$ and let $\{y_\alpha\}$ be a point in W such that $\Pi_\alpha(y_\alpha) = y_\alpha$. Now since W is θ-open, there exists a basic open set $U = U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1 \cdots \alpha_n} X_\alpha$ such that $\{y_\alpha\} \in U \subset \mathrm{Cl}(U) = \mathrm{Cl}(U_{\alpha_1}) \times \mathrm{Cl}(U_{\alpha_2}) \times \cdots \times \mathrm{Cl}(U_{\alpha_n}) \times \prod_{\alpha \neq \alpha_1 \cdots \alpha_n} X_\alpha \subset W$. Without loss of generality, we may assume that for some $1 \leq j \leq n$, $\alpha = \alpha_j$. Thus, $y_\alpha \in \Pi_\alpha \mathrm{Cl}(U) \mathrm{Cl}(U_{\alpha_j}) \subset \Pi_\alpha(W)$ so that each point of $\Pi_\alpha(W)$ contains a closed neighborhood lying in $\Pi_\alpha(W)$. It follows that $\Pi_\alpha(W)$ is θ-open.

THEOREM 7. *Let $f: X \to Y$ be a function from X onto Y that is both open and closed. Then f preserves θ-open sets.*

PROOF. Let U be θ-open in X and let $y \in f(U)$. Then there exists an $x \in U$ such that $f(x) = y$ and an open set U_0 such that $x \in U_0 \subset \mathrm{Cl}(U_0) \subset U$. Therefore, $f(x) = y \in f(U_0) \subset f(\mathrm{Cl}(U_0)) \subset f(U)$. Now, the fact that f is both open and closed shows that $f(U_0)$ is an open set whose closure $\mathrm{Cl}(f(U_0)) \subset \mathrm{Cl}(f(\mathrm{Cl}(U_0))) = f(\mathrm{Cl}(U_0))$ is contained in $f(U)$. This shows that $f(U)$ is θ-open.

THEOREM 8. *Let $f: X \to Y$ be continuous. If $V \subset Y$ is θ-open, then $f^{-1}(V)$ is θ-open in X.*

PROOF. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists an open set U such that $f(x) \in U \subset \mathrm{Cl}(U) \subset V$ because V is θ-open. Thus, $x \in f^{-1}(U) \subset f^{-1}(\mathrm{Cl}(U)) \subset f^{-1}(V)$. The continuity of f then gives $f^{-1}(U)$ as an open set whose closure is contained in $f^{-1}(V)$ which shows that $f^{-1}(V)$ is θ-open.

3. Faintly-continuous functions

DEFINITION 2. Let X and Y be topological spaces. Then $f: X \to Y$ is *faintly-continuous* if for each $x \in X$ and θ-open V containing $f(x)$, there exists an open set U containing x such that $f(U)\subset V$.

As will be demonstrated shortly, the concept of faintly-continuous is a very weak form of continuity. Perhaps the concept could have been better named θ-continuous, but that notation is already reserved for a different kind of non-continuous function. (See, for example, Definition 2 of [6].)

THEOREM 9. *Let $f: X\to Y$ be given. Then they are equivalent:*

(a) $f: X\to(Y,T)$ is faintly-continuous.

(b) $f: X\to(Y,T_\theta)$ is continuous.

(c) The inverse image of each θ-open set in (Y,T) is open in X.

(d) The inverse image of each θ-closed set in (Y,T) is closed in X.

PROOF. The implications follow easily from the definitions.

A function $f: X\to Y$ is called *weakly-continuous* [4] if for each $x\in X$ and each open set V containing $f(x)$ there exists an open set U containing x such that $f(U)\subset \mathrm{Cl}(V)$.

THEOREM 10. *If $f: X\to Y$ is weakly-continuous, then f is faintly-continuous.*

PROOF. Let $x\in X$ and let V be a θ-open set containing $f(x)$. Then there exists an open set W such that $f(x)\in W\subset \mathrm{Cl}(W)\subset V$. Now, since f is weakly-continuous, there exists an open set U containing x such that $f(U)\subset \mathrm{Cl}(W)\subset V$. Consequently f is faintly-continuous.

EXAMPLE 2. A faintly-continuous function need not be weakly-continuous. Let $X=\{0,1\}$ with topology $\{\phi, X, \{1\}\}$ and let $Y=\{a,b,c\}$ with topology $\{\phi, Y, \{a\}, \{b\}, \{a,b\}\}$. Finally, let $f: X\to Y$ be defined as $f(0)=a$ and $f(1)=b$. Then f is not weakly-continuous at $x=0$, but f is faintly-continuous since the only θ-open set in Y is Y itself.

Theorem 10 and Example 2 now allow us to see the position faintly-continuous functions occupy among other well-known non-continuous functions. First, however, we should recall the definitions of almost-continuity and θ-continuity: a function $f: X\to Y$ is *almost-continuous* (θ-continuous) if for each $x\in X$ and each regular-open V (open V) containing $f(x)$, there exists an open U containing x such that $f(U)\subset V$ ($f(\mathrm{Cl}(U))\subset \mathrm{Cl}(V)$). Now it readily follows that

continuity$\Rightarrow$almost-continuity$\Rightarrow\theta$-continuity$\Rightarrow$weak-continuity$\Rightarrow$faint-continuity.

These implications, aside from the last one, are explored in [6].

THEOREM 11. *Let (Y,T) be an almost-regular space and $f: X\to(Y,T)$ a* *faintly-continuous function. Then f is almost-continuous.*

PROOF. Since $f: X \to (Y, T)$ is faintly-continuous, then $f: X \to (Y, T_\theta)$ is continuous. But (Y, T) almost-regular implies $T_\theta = T_s$ by the Corollary to Theorem 3. Thus, $f: X \to (Y, T_s)$ is continuous showing that $f: X \to (Y, T)$ is almost-continuous.

COROLLARY TO THEOREM 11. *If (Y, T) is almost-regular and $f: Y \to (Y, T)$, then they are equivalent:*

(a) f is faintly-continuous.

(b) f is weakly-continuous.

(c) f is θ-continuous.

(d) f is almost-continuous.

In the above Corollary, if almost-regular is replaced with regular, then we may add continuity to the list of equivalences.

THEOREM 12. *If $f: X \to Y$ is faintly-continuous and $A \subset X$, then $f|A: A \to Y$ is faintly-continuous.*

PROOF. Evident.

For a given $f: X \to Y$, the graph map $g: X \to X \times Y$ is defined as $g(x) = (x, f(x))$.

THEOREM 13. *If the graph map of $f: X \to Y$ is faintly-continuous, then f is faintly-continuous.*

PROOF. Let $x \in X$ and let V be θ-open in Y containing $f(x)$. Then $X \times V$ is θ-open in $X \times Y$ by Theorem 5 and contains $g(x) = (x, f(x))$. Since the graph map $g: X \to X \times Y$ is faintly-continuous, there exists an open set U containing x such that $g(U) \subset X \times V$. This implies that $f(U) \subset V$ so that f is faintly-continuous.

THEOREM 14. *If $f: X \to Y$ is weakly-continuous, then the graph map $g: X \to X \times Y$ is faintly-continuous.*

PROOF. Let $x \in X$ and let W be a θ-open set containing $g(x)$. Then there is a closed neighborhood, hence a closed basic open set $\mathrm{Cl}(U \times V)$, containing $g(x)$ and lying inside W. Thus, $g(x) = (x, f(x)) \in \mathrm{Cl}(U \times V) = \mathrm{Cl}(U) \times \mathrm{Cl}(V)$ so that $f(x) \in \mathrm{Cl}(V)$. Since f is weakly-continuous, there exists an open set $U_0 \subset U$ containing x such that $f(U_0) \subset \mathrm{Cl}(V)$. Consequently, $g(U_0) \subset \mathrm{Cl}(U) \times \mathrm{Cl}(V) \subset W$ showing g to be faintly-continuous.

3. Functions with extremely-closed graphs

DEFINITION 3. The graph $G(f)$ of $f: X \to Y$ is *extremely-closed* if for each $(x, y) \notin G(f)$ there exists an open set U containing x and a θ-open set V containing y such that $(U \times V) \cap G(f) = \phi$.

The proofs to the next two theorems follow easily from the above definition.

THEOREM 15. *The graph of $f: X \to Y$ is extremely-closed if and only if for each $x \in X$ and $y \neq f(x)$ there exists an open set U containing x and a θ-open set V containing y such that $f(U) \cap V = \phi$.*

THEOREM 16. *The graph of $f: X \to (Y, T)$ is extremely-closed if and only if the graph of $f: X \to (Y, T_\theta)$ is closed.*

THEOREM 17. *If $f: X \to (Y, T)$ is faintly-continuous and (Y, T_θ) is Hausdorff, then f has an extremely-closed graph.*

PROOF. We know that $f: X \to (Y, T_\theta)$ is continuous because $f: X \to (Y, T)$ is faintly-continuous. Since T_θ is Hausdorff, the graph of $f: X \to (Y, T_\theta)$ is closed [2, Theorem 1,5(3), p. 140]. Thus, $f: X \to (Y, T)$ has an extremely-closed graph by Theorem 16.

THEOREM 18. *Let Y be completely Hausdorff and let $f: X \to Y$ be faintly-continuous. Then $G(f)$ is extremely-closed.*

PROOF. Let $x \in X$ and let $y \neq f(x)$. Since Y is completely Hausdorff, there exists a continuous $g: Y \to R$ such that $g(f(x)) \neq g(y)$. Thus, there exist open disjoint sets W and G containing $g(f(x))$ and $g(y)$, respectively, such that $g^{-1}(W) \cap g^{-1}(G) = \phi$. But $g^{-1}(W)$ is θ-open by Theorem 8 and the fact that every open subset of R is θ-open. Therefore, there exists an open U containing x such that $f(U) \subset g^{-1}(W)$ so that $f(U) \cap g^{-1}(G) = \phi$. Theorem 15 now implies that the graph of f is extremely-closed.

The graph of $f: X \to Y$ is called *strongly-closed* [5] if for each $(x, y) \notin G(f)$ there exist open sets U and V containing x and y, respectively, such that $(U \times \mathrm{Cl}(V)) \cap G(f) = \phi$.

THEOREM 19. *Let $f: X \to Y$ have an extremely-closed graph. Then f has a strongly-closed graph.*

PROOF. Let $x \in X$ and $y \neq f(x)$. Then by Theorem 15, there exists an open set U containing x and a θ-open set V containing y such that $f(U) \cap V = \phi$. Since V is θ-open, there exists an open set V_0 such that $y \in V_0 \subset \mathrm{Cl}(V_0) \subset V$ so that $f(U) \cap \mathrm{Cl}(V_0) = \phi$. It follows that the graph of f is strongly-closed by the first Lemma of [7].

From Theorem 19 and [5] we now see the position of extremely-closed graphs as follows:

extremely-closed graph $\Rightarrow$ strongly-closed graph $\Rightarrow$ closed graph. It is shown in [5] that a closed graph need not be strongly-closed. Our last example shows the first implication above cannot, in general, be reversed.

EXAMPLE 3. Let $Y = [0, 2)$ and let G_k be defined by

$$G_k = \bigcup \left\{ \left(\frac{2n+1}{2n(n+1)}, \frac{2n-1}{2n(n-1)} \right) : n > k,\ n \text{ is odd} \right\},\ k \in N.$$

Let H_k be defined as in Example 1 and topologize Y using the following subbasic open sets: $\{V \subset Y - \{1\} : V \text{ open in } R\} \cup \{H_k \cup G : k \in N,\ G \subset Y,\ G \text{ open in } R$ and contains the point $1\} \cup \{G_k \cup 0 : k \in N\}$. Now define $f : X \to Y$ by $f(x) = x$ for all $x \in X$ where X is the space given in Example 1. Then f is continuous and Y is Hausdorff which implies $G(f)$ is strongly-closed by the Corollary to Theorem 1 of [5]. However, the point $(1, 0) \notin G(f)$, but for each open U containing 1 and each θ-open set V containing 0, $(U \times V) \cap G(f) \neq \phi$. Therefore, $G(f)$ is not extremely-closed.

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