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ISOMETRIC EMBEDDINGS OF LORENTZIAN MANIFOLDS BY THE SOLUTIONS OF THE D'ALEMBERTIAN EQUATION

By Jong-Chul Kim

Introduction

The topological and proper embeddings of Lorentzian manifolds by solutions of the d'Alembertian equations were investigated in [6]. In this paper we shall show that Lorentzian manifolds can be isometrically embedded into a pseudo-euclidean space of suitable dimension.

By taking the covariant and contravariant derivatives of a scalar field, the divergence of the gradient of a scalar field becomes an invariant second order linear partial differential operator,

$$\Box u=\nabla_i\nabla^i u=\|g\|^{-\frac{1}{2}}\frac{\partial}{\partial x_i}\left(\|g\|^{\frac{1}{2}}g^{ij}\frac{\partial u}{\partial x_i}\right)$$

where g is a Lorentzian metric, $|g|$ determinant of g, $\|g\|$ absolute value of $|g|$, and g^{ij} is a component of the inverse of the matrix (g_{ij}) consisting of components of the metric tensor g. The d'Alembertian equation means that

$$\Box u=0.$$

The function u could be a vector valued function, and the equation, in this case, consists of a system of the system above. In any cases, solutions of the equation are considered in the causal domain, and we need to treat the equation under the causal conditions and global hyperbolicity for the global solutions.

We showed the fact that any point in smooth Lorentzian manifold has a coordinate chart whose functions consist of solutions of the d'Alembertian equation in [6]. We will call this kind of the chart "d'Alembertian coordinate chart" throughout this paper. We assume that the manifold is smooth and globally hyperbolic Lorentzian manifold of dimension m. IR^p denotes a Euclidean space of dimension p, and IR^p_q a pseudo-euclidean space of p-positive and q-negative signatures. $g(u)$ denotes the induced metric of a scalar function u. We shall need to consider sums and direct products of embeddings defined by

$$(u_1+u_2)(x)=u_1(x)+u_2(x),\ \ (u_1\times u_2)(x)=(u_1(x),\ u_2(x))$$

where u_i's are functions of M to IR^p_q.

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Clearly, $g(u_1 \times u_2) = g(u_1) + g(u_2)$.

Metrics on M will be partially ordered by setting

$$g_1 \leq g_2 \text{ iff } g_1(X, X) \leq g_2(X, X) \text{ for } X \text{ in the tangent bundle } TM.$$

u_* means the differential mapping of the mapping u. Other terminologies refer to the references.

2. Isometric embeddings

The following lemmas have been shown in [6].

LEMMA(A). *For each point in M there exists a coordinate neighborhood of the point in causal domain whose coordinate functions consist of solutions of the d'Alembertian equation.*

LEMMA(B). *If a manifold M is smooth and globally hyperbolic Lorentzian manifold of dimension m, then M can be topologically (properly) embedded into IR^{2m+1} and also $IR_1^{p+1}(p \geq 2m+1)$ by solutions of the d'Alembertian equation.*

LEMMA(C). *If g is a Lorentzian metric, then there exists a mapping u of M to IR_1^1 such that $g(u) \leq g$ and the components of u consist of solutions of the d'Alembertian equation.*

LEMMA(D). *If g is a positive definite metric on M, then there is a topological (proper) embedding u of M into IR_1^{2m+1} and also IR_1^{p+1} $(p \geq 2m+1)$ such that $g(u) \leq g$ and the components of u consist of solutions of the d'Alembertian equation.*

Let u be an embedding of M to IR^p. Let X be a normal basis of local cross section of the normal bundle ν over $u(M)$.

Define $F : TM \longrightarrow \nu$ by $F(A) = \Sigma_\alpha X_\alpha \eta(X_\alpha, X_\alpha)\eta(X, \nabla_{u_* A} u_* A)$, where ∇ is covariant differentiation in IR^p and η is metric tensor on IR^p. If F is everywhere two-one, we call u perturbale. A perturbale embedding is locally one such that the mapping from ν to the space of symmetric covariant tensor on M given by

$$w : \xi \longrightarrow -2\eta_{\alpha\beta} \frac{\partial^2 z^\alpha(f(x))}{\partial x^a \partial x^b} \xi^\beta dx^a dx^b$$

is onto, where x^a is coordinates in an open set in M and z^α is coordinates in IR^p. We adopt the C^k-topology on the bundle of symmetric covariant second rank tensors over M as usual way, where k is larger than 3 and ∞.

LEMMA 1. *If C^{k+2}-mapping $V : M \longrightarrow IR^p$ is perturbale and components of v*

are solutions of the d'Alembertian equation, then there exists a C^k-neighborhood U of $g(v)$ such that for g in U there is a perturbale embedding $u: M \longrightarrow IR^p$ with $g(u)=g$.

PROOF. Since M is globally hyperbolic, we can take increasing compact subsets and necessary subsets of the compact subsets of M, in order to smooth the relative functions and get our perturbale embedding whose components are solutions of the d'Alembertian equation as Clarke [1] and Nash [11] did. Our functions are different from theirs. However, applying the global hyperbolicity of M to solutions of the equation, the method is similar to them. The detailed proof is omitted.

LEMMA 2. *If g is a positive definite metric on M, there exists an embedding $u: M \longrightarrow IR^p$ such that u is perturbale, $g(u) \leq g$ and the components of u are solutions of the d'Alembertian equation on M.*

PROOF. By Lemma A and D, there exists an immersion u of M into IR^{2m} such that $g(u) \leq g$ and components of u are solutions of the d'Alembertian equation.

Let $v^k = D^k_{ij} y^i y^j$ and y^i be coordinates in IR^{2m}, where the D^k are constant symmetric $2m \times 2m$-matrices and $k=1, 2, \cdots, p$. Choose p-dimensional subspace L of set of all such matrices such that L is spanned by the D^k's. For the d'Alembertian coordinates x^a at x in M the map $h_x : D^k_{ij} \longrightarrow \partial^2 v^k / \partial x^a \partial x^b$ is linear from K into symmetric tensor space of rank two at x. Then it is clear that, if the dimension of the intersection of L with the kernel of h_x is the whole of this space.

Applying it to each x in M and using an immersion u, the set of h_x such that $h_x | L$ is not onto is of dimension q, where q is less than or equal to $p(2m^2 + m - p - 1) + \frac{1}{2} m(m+3) - 1$. Thus, by adjusting the dimension $p \geq \frac{1}{2} m(m+3)$, we can choose L on which h_x is onto for all x in M. This means that any immersion of the form

$$C^k_i v^i + D^k_{ij} v^i v^j \left(k=1, 2, \cdots, \frac{1}{2} m(m+5)\right)$$

will be perturbale, where C's are arbitrary. By taking u to satisfy the intersec ting condition for a regular immersion, our lemma follows as well as $g(u) \leq g$.

LEMMA 3. *If M is globally hyperbolic, then M can be isometrically embedding* *into* IR^p, $p\geq\frac{1}{6}m\ (2m^2+37)+\frac{5}{2}m^2+1$, *by solutions of the d'Alembertian equation on* M.

PROOF. This lemma can be proved by using Lemma A, C, Lemma 1 and 2. We will omit the detailed proof.

LEMMA 4. *If* M *is smooth and globally hyperbolic Lorentzian manifold with the metric* g *and dimension* m, *there exists a mapping* $u: M\longrightarrow IR_1^1$ *such that the components of* u *are solutions of the d'Alembertian equation on* M *and* $g(u)\leq g$.

PROOF. Let u_1 be a mapping of M to IR_1^1 whose components are solutions of the d'Alembertian equation. Take a covering of M as consisting of the d'Alembertian coordinates charts

$$(U_i,\ (x^1, x^2, \cdots, x^m))$$

with $\frac{\partial}{\partial x^m}$ time-like and $u_{1*}\left(\frac{\partial}{\partial x^m}\right)\neq 0$ in a small neighborhood of x in M.

Let μ be a smooth mapping of IR^1 to IR^2 by

$$\mu(x)=(x, p_n(x))$$

such that $\|d\mu\|=$constant, $p=$smooth periodic function with integer period a and $\mu(na)=(na, 0)$.

Let $\{\varphi_i\}$ be a smooth partion of unity subordinate to the above covering and

$$u_1(x)=u_1(x^i, x^m),\ i=1, 2, \cdots, m-1.$$

Define a mapping u of M to IR_1^1 by

$$u(x)=\begin{cases}u_1(x) \text{ if } x\in U_1\\ u_1(x^i, x^m(1-\varphi_1)+\varphi_1\mu_1(x^m))+(0,1)\mu^2(x^m)\varphi_1, \text{ otherwise,}\end{cases}$$

where μ_i is component of μ.

Using now Kuiper's method [8] and approaching n to infinity, the lemma can be complete, that is $g(u)\leq g$.

THEOREM. *If a smooth manifold* M *of dimension* m *is globally hyperbolic, then* M *can be isometrically embedded into* IR_q^p *by solutions of the d'Alembertian equation on* M, *where* $q\geq 1$, *and* $p\geq\frac{1}{6}m(2m^2+37)+\frac{5}{2}m^2+2$.

PROOF. Let g be the Lorentzian metric such as $g=g_1+g_2$ where g_1 is of rank

m and signature $m-2$, and g_2 is positive.

Let u_1 be the mapping given by Lemma 4 such that $g(u_1) \leq g_1$. Now, let u_2 be the embedding given by Lemma 3 such that $g(u_2) = g_1 - g(u_1) + g_2$.

Then, letting $u = u_1 \times u_2$, u is the isometric embedding of M to IR^p_q.

Young-Nam University, Daegu, Korea.

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