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# INVARIANCE OF DIMENSIONS BETWEEN AFFINE VARIETY AND ITS PROJECTIVE CLOSURE

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## 0. Introduction

It is well-known fact that the dimension of an affine variety is equal to that of its affine coordinate ring. But, this is no true for the case of projective variety.

In this paper, I shall prove that the dimension of a projective variety is equal to that of its homogeneous coordinate ring minus 1.

Furthermore, using this fact, I shall show that, if we identify an affine variety with a subset of a projective space, the dimension of this set is always equal to that of its projective closure.

## 1. Preliminaries

Throughout this paper, we fix $k$ an algebrajcally closed field.

An a *ffine n-space* $A^n$ over $k$ is the set of all $n$-tuples of elements of $k$,

If $T$ is any subset of $k[x_1 \cdots, x_n]$, the ***zero set*** of $T$ is the set

$$Z(T)=\{ a\in A^n : f(a)=0 \text{ for all } f\in T \}.$$

A subset $Z$ of $A^n$ is an ***algebraic set*** if there exists a subset $T$ in $k[x_1, \cdots, x_n]$ such that

$$Z=Z(T).$$

Then, $A^n$ becomes a topological space by taking the closed subsets to be the algebraic sets. This topology is the ***Zarisky topology***.

A nonempty subset of a topological space is ***irreducible*** if it cannot be expressed as the union of two nonempty proper closed subsets. If $S$ is irreducible, the closure $S$ is also irreducible.

For any subset $\bar{S}$ in $A^n$, the ***ideal*** of $S$ in $k[x_1, \cdots, x_n]$ is the set

$$I(S)=\{ f\in k[x_1, \cdots, x_n] : f(a)=0 \text{ for all } a\in S \}$$

There is a ono-to-one inclusion-reversing correspondence between algebraic sets in $A^n$ and residue ideals in $k[x_1, \cdots, x_n]$. Moreover, an algebraic set is irreducible iff its ideal is a prime ideal. If $Z\subset A^n$ is an affine algebraic set, we define the ***affine coordinate ring*** $\Gamma(Z)$ of $Z$ to be $k[x_1, \cdots, x_n] / I(Z)$.

The ***projective n-space*** $P^n$ over $k$ is the set of all equivalence classes if $A^{n+1} - \{0\}$ under the relation given by $(a_0, \cdots, a_n) = (b_0, \cdots, b_n)$ if there exists $t \in k^*$, the multiplicative set of $k$, such that $b_i = ta_i$ for all $i$. Denote by $a$ the point in $P^n$ which is represented by $a \in A^{n+1} - \{0\}$. This $a$ in $A^{n+1} - \{0\}$ is called a ***homogeneous coordinate*** for $a$ in $P^n$. If $T$ is any set of homogeneous elements of $k[x_0, \cdots, x_n]$, the ***zero set*** of $T$ is the set

$$V(T) = \{ a \in P^n : f(a) = 0 \text{ for all } f \in T \}$$

A ***homogeneous ideal*** in $k[x_0, \cdots, x_n\}$ is the ideal generated by homogeneous elements in $k[x_0, \cdots, x_n]$.

If $\mathfrak{a}$ is a homogeneous ideal of $k[x_0, \cdots, x_n]$ with generator $T$ of homogeneous elements, set $V((\mathfrak{a}) = V(T)$. A subset $V$ of $P^n$ is an ***algebraic set*** if there exists a homogeneous ideal $\mathfrak{a}$ such that $V = V(\mathfrak{a})$. We define the ***Zariski topology*** on $P^n$ by taking the closed sets to be the algebraic sets. A ***projective variety*** is an irreducible algebraic set in $P^n$. An open subset of projective variety is a ***quasi-projective variety***. If $S$ is any subset of $P^n$, we define the ***homogeneous ideal*** $I(S)$ of $S$ in $k[x_0, \cdots, x_n]$ to be the ideal generated by

$$\{f \in k[x_0, \cdots, x\ ] : f \text{ is homogeneous and } f(a) = 0 \text{ for all } a \in S\}$$

If $V$ is an algebraic set, the ***homogeneous coordinate ring*** of $V$ is the ring $\Gamma(V) = k[x_0, \cdots, x_n] / I(V)$.

Let $G_d$ be the group of homogencous polynomials of $k[x_0, \cdots, x_n]$ with degree d. Then, $k[x_0, \cdots, x_n] = {}^{\oplus} d \geqslant O G_d$. The ***irrelevant maximal ideal*** is the homogeneous ideal ${}^{\oplus} d > O G_d$. There is one-to-one inclusion-reversing correspondence between algebraic sets in $P^n$ and homogeneous reduced ideals in $k[x_0, \cdots, x_n]$ not equal to the irrelevant maximal ideal. Furthermore, an algebraic set is irreducible iff its ideal is a prime ideal.

If $X$ is a topological space, we define the ***dimension*** of $X$ to be the supremum of all integers $n$ such that there exists a chain

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

of distinct irreducible closed subsets of $X$. Denote this by dim $X$.

**Lemma 1.** Let $X$ be a topological space of finite dimension. If $\{U_i\}$ is an open covering of $X$, then, dim $X = \sup \dim U_i$.

**Proof.** Since the closure of an irreducible set is also irreducible, sup dim $U_i$

$\leq dim\ X$. Suppose sup $dim\ U_i < dim\ X = n$.

Let $X_0 \subset X_1 \subset \cdots \subset X_n$ be a maximal chain of distinct closed irreducible subsets in $X$. Then, for each $i$, $X_0 \cap U_i \subset X_1 \cap U_i \subset \cdots \subset X_n \cap U_i$ is a chain of closed irreducible subsets in $U_i$. Since dim $U_i < n$, $X_{k-1} \cap U_i = X_k \cap U_i$ for some $k$.

But then, we have

$$\begin{aligned} X_k &= X_{k-1} \cup (X_k - X_{k-1}) \\ &= X_{k-1} \cup [X_k - (X_{k-1} \cap U_i)] \\ &= X_{k-1} \cup [X_k - (X_k \cap U_i)] \end{aligned}$$

Since $X_{k-1}$ is a closed proper subset of $X_k$ and since $X_k - (X_k \cap U_i)$ is closed in $X_k$, we have $X_k = X_k - (X_k \cap U_i)$ from the irreducibility of $X_k$. Hence, $X_k \cap U_i = \phi$.

Since $X_0 \subset X_k$, $X_0 \cap U_i = \phi$. This is true for all $i$, and hence, $X_0 \cap (\bigcup_i u_i) = \phi$.

Since $U_i$ is an open covering of $X$, we have $X_0 = \phi$, which is a contradiction.

If $A$ is a commutative ring, the ***height*** of a prime ideal is the supremum of all integers $n$ such that there exists a chain

$$P_0 \subset P_1 \subset \cdots \subset P_n = P$$

of distinct prime ideals. Denote this by ht$P$.

We define the ***dimension*** of $A$ to be the supremum of the heights of all prime ideals. Denote this by dim $A$. If a finitely generated k-algebra $A$ is an integral domain, dim $A$ is equal to the transcendental degree of the quotient field $K(A)$ of $A$ over $k$.

For any prime ideal $P$ in $A$, we have

$$ht\ P + dim\ A/P = dim\ A.$$

From the definition, dim $A_p = ht\ P$ for any prime ideal.

**Note.** If $A = k[x_1 \cdots, x_n]$, dim $A = n$.

If $Z$ is an affine variety in $A^n$, $I(Z)$ is prime. Hence, dim$\Gamma(Z) = n - ht\ I(Z)$.

If $V$ is a projective variety in $P^n$,

$$dim \Gamma(V) = n + 1 - ht\ I(V).$$

## 2. Dimensions of Projective Variety

It is well-known fact that $dim\ Z = dim \Gamma(Z)$ if $Z$ is an affine variety. But this is not true for the case of projective variety. In fact, we have $dim\ V = dim\ \Gamma(V) - 1$ where $V$ is a projective variety. Now, we investigate this fact.

If $f \in k[k_o, \cdots, x_n]$ is a linear homogeneous polynomial, then the zero set of $f$ is called a ***hyperplane***. We denote the zero set of $X_i$ by $H_i$ for $i=0, 1, \cdots, n$. Let $A_i^n$ be the open set $P^n - H_i$. Then, $P^n$ is covered by these open sets $A_i^n$. Define $h_i : A_i^n \to A^n$ as follows; if $a=(a_o, \cdots, a_n) \in A_i^n$, $h_i(a)=(a_o/a_i, \cdots, a_n/a_i)$ with $a_i/a_i$ omitted. The map $h_i$ is a homeomorphism of $A_i^n$ with its induced topology to $A^n$ with its Zariski topology.

Let $G^h$ be the set of homogeneous elements of $k[x_o, \cdots, x_n]$. We define a map $\varphi : G^h \to k[x_1, \cdots, x_n]$ and a map $\psi : k[x_1, \cdots, x_n] \to G^h$ by following: Given $f \in G^h$, $\varphi(f) = f(1, x_1, \cdots, x_n)$ and given $g \in k[k_1, \cdots, x_n]$, $\psi(g) = x_o^d g(x_1/x_o, \cdots, x_n/x_o)$ where $d$ is the degree of $g$.

**Lemma 2** Let $V$ be a projective variety and $Z_i$ be the affine variety $h_i(V \cap A_i^n)$ of each $i$. Then, $\Gamma(Z_i)$ can be identified with the subring of elements of degree $O$ of the localized ring $\Gamma(V)_{x_i}$.

**Proof**. Without loss of generality, we may ausume $x_i = x_o$.

If $g \in \Gamma(Z_o)$, then $f=\psi(g)$ is an element of $\Gamma(V)$. Indeed, if $g \in I(Z_o)$, it is enough to show that $f \in I(V)$.

Let $a \in V$ be given. Then, $(a_1/a_0, \cdots, a_n/a_o) \in Z_o$. Hence, $g(a_1/a_o, \cdots, a_n/a_o) = 0$ since $g \in I(Z_o)$. Then, $f(a) = a_o^d g(a_1/a_o, \cdots, a_n/a_o) = 0$. This shows that $f \in I(V)$.

Now, the assignment $g \in \Gamma(Z_o) \to f/x_o^d \in \Gamma(V)x_o$ is well-defined homomorphism. If $f/x_o^d$, $f \in \Gamma(V)$, is an element of degree $O$ in $\Gamma(V)x_o$, $f$ must be a homogeneous polynomial of degree $d$ in $k[x_o, \cdots, x_n]$. Then, $g=\psi(f)$ is an element of $\Gamma(Z_o)$, which shows that the mapping is surjective. If $f/x_o^d = O$ in $\Gamma(V)x_o$, $f(a_0, a_1, \cdots, a_n)=0$ for all $(a_o, \cdots a_n) \in V$. Hence, for every $(a_1, \cdots, a_n) \in Z_o$, $f(1, a_1, \cdots, a_n)=0$, since $(1, a_1, \cdots, a_n) \in V$.

This shows that $g(a_1, \cdots, a_n)=0$. Thus, $g \in I(Z_o)$, which shows that the mapping is injective.

**Lemma 3**. Under the same assumption in lemma 2,

$$\Gamma(V)x_i = \Gamma(Z_i)[x_i, x_i^{-1}]$$

**Proof**. It is clear that $\Gamma(Z_i)[x_i, x_i^{-1}] \subset \Gamma(V)x_i$. Conversely, suppose $f/x_i^e \in \Gamma(V)x_i$.

Let $f = f_d + f_{d-1} + \cdots + f_o$, where $f_k$ is the homogencous polynomial of degree $k$.

Then, $f_k/x_i^k \in \Gamma(Z_i)$. Hence, $f/x_i^e = \sum_{k=0}^{d} (f_k/x_i^k) x_i^{k-e}$ is an element of $\Gamma(Z_i)[x_i, x_i^{-1}]$.

**Theorem 1.** If $V$ is a projective variety, then $\dim \Gamma(V) = \dim V + 1$.

**Proof**. By Lemma 1, $\dim V = \dim Z_i$ for some $i$. Then $Z_i \neq \phi$, hence, $(x_i)$ is a maximal ideal of $\Gamma(V)$. Thus,

$$\dim \Gamma(V)_{x_i} = ht\ (x_i) = \dim \Gamma(V).$$

Since $\Gamma(Z_i)$ and $\Gamma(Z_i)[x_i, x_i^{-1}]$ are finitely generated $k$-algebra which are integral domains, we have

$$\dim \Gamma(Z_i) = tr.\ deg.\ k(\Gamma(Z_i)) \text{ and}$$

$$\dim \Gamma(Z_i)[x_i, x_i^{-1}] = tr.\ deg.\ k(\Gamma(Z_i)[x_i, x_i^{-1}]).$$

But, since tr. deg. $K(\Gamma(Z_i)[x_i, x_i^{-1}]) = tr.\ deg.\ K(\Gamma(Z_i)) + 1$ and Lemma 3, we have

$$\dim \Gamma(V)_{x_i} = \dim \Gamma(Z_i) + 1.$$

Since $Z_i$ is an affine variety, $\dim \Gamma(Z_i) = \dim Z_i$. Consequently, $\dim \Gamma(V) = \dim Z_i + 1 = \dim V + 1$.

### 3. Projective Closure of an Affine Variety

We identify $A^n$ with the open set $A_0^n$ in $P^n$ by the hoeomorphism $h_0$. If $Z$ is an affine variety in $A^n$, we can speak of the closure of $Z$ in $P^n$ with this identification. This closure is called the *projective closure* of the affine variety $Z$.

**Lemma 4.** Let $Z$ be an affine variety in $A^n$ and $V$ be its projective closure in $P^n$. Then

$$ht\ I(Z) = ht\ I(V).$$

**Proof**. It suffices to show that $I(V)$ is the ideal generated by $\psi(1\ (Z))$.

We first show that $\psi(I(Z)) \subset I(V)$.

If $f \in \psi(I(Z))$, $f = \psi(g)$ for some $g \in I(Z)$. Let $d$ be the degree of $g$, then

$$f = x_0^d g(x_1/x_0, \cdots, x_n/x_0).$$

For given $a \in V$, $(a_1/a_0, \cdots, a_n/a_0) \in Z$. Hence, $g(a_1/a_0, \cdots, a_n/a_0) = 0$. Thus, $f(a) = a_0^d g(a_1/a_0, \cdots, a_n/a_0) = 0$. Hence, $f \in I(V)$.

Next, we show that $I(V)$ is the ideal generated by $\psi(I(Z))$.

Suppose that $\mathfrak{a}$ is any homogeneous reduced ideal containing $\psi(I(Z))$. We shall show that $I(V)\subset \mathfrak{A}$, and it is enough to show that $Z(\mathfrak{A})\subset V$. Let $(a_0, \cdots, a_n)\in Z(\mathfrak{A})$, then $f(a_0, \cdots, a_n)=0$ for all $f\in \mathfrak{A}$.

Since $\psi(I(Z))\subset \mathfrak{A}$. for every $g\in I(Z)$, we have

$$(\psi g)\ (a_0, \cdots, a_n)=a_0^d g(a_1/a_0, \cdots, a_n/a_0)=0.$$

Hence, $(a_1/a_0, \cdots, a_n/a_0)\in Z(I(Z))=Z$ in $A^n$. Thus $(a_0, \cdots, a_n)\in Z\subset V$ in $P^n$.

**Remark.** Although $f_1, \cdots, f_r$ generate $I(Z)$, $\psi(f_1), \cdots, \psi(f_r)$ do not necessarily generate $I(V)$.

For example, let $Z\subset A^n$ be the set

$$Z=\{\ (t, t^2, t^3)\ :t\in k\ \}$$

Then, $f_1=x_2-x_1^2$, $f_2=x_3-x_1^3$ generate $I(Z)$. But, $\psi(f_1)=x_0x_2-x_1^2$, $\psi(f_2)=x_0^2x_3-x_1^3$ do not generate $I(V)$.

Indeed, $x_0x_3-x_1x_2\in I(V)$, but it is not any combination of $\psi(f_1)$ and $\psi(f_2)$.

**Theorem 2.** Under the same assumption in Lemma 4, we have

$$dim\ V=dim\ Z.$$

**Proof**: $dim\ Z=dim\Gamma(Z)$

$$=n-ht\ I(Z)$$

$$=(n+1-ht\ I(V))-1$$

$$=dim\Gamma(V)-1$$

$$=dim\ V.$$

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