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CHARACTERIZATIONS OF QUASI *H*-CLOSED SPACES

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1. Introduction and preliminaries

In [6], J. Porter and J. Thomas introduced the concept of quasi H-closed spaces. We give characterizations of such spaces and sets quasi H-closed relative to the space, some of which make use of a type of convergence we define as w-convergence.

Throughout this paper, X means a topological space on which no separation axioms are assumed. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively. A subset A of X is said to be *regular open* (resp. *regular closed*) [2] if $A=\mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A=\mathrm{Cl}(\mathrm{Int}(A))$). A set A of X is said to be *semiopen* [1] [5] (resp. *regular semiopen* [1]) if there exists an open (resp. regular open) set U such that $U\subset A\subset \mathrm{Cl}(U)$. A space X is said to be *extremally disconnected* [5] if the closure of every open set is open. A space X is said to be *nearly compact* [3] if every open cover has a finite subfamily, the interior of the closures of which cover X, and *RS-compact* [3] (resp. *S-closed* [1], [4]) if every regular semiopen (resp. semiopen) cover has a finite subfamily whose interiors (resp. closures) cover X.

DEFINITION 1.1. A topological space X is *quasi H-closed* (denoted QHC) [1] if every open cover has a finite proximate subcover (every open cover has a finite subfamily whose closures cover the space).

DEFINITION 1.2. A filterbase $\mathfrak{F}=\{A_i\}$ *w-converges* to a point $x\in X$ [cf. 2, 5] if for each open set V containing x, there exists an $A_i\in\mathfrak{F}$ such that $A_i\subset \mathrm{Cl}(V)$.

DEFINITION 1.3. A filterbase $\mathfrak{F}=\{A_i\}$ *w-accumulates* to a point $x\in X$ [cf 2, 5] if for each open set V containing x and $A_i\in\mathfrak{F}$, $A_i\cap \mathrm{Cl}(V)\neq\phi$.

We now obtain an easy consequence of these definitions whose proof is omitted.

THEOREM 1.1. *Lef $\mathfrak{F}$ be a maximal filterbase in X. Then $\mathfrak{F}$ w-accumulates to a point $x\in X$ if and only if $\mathfrak{F}$ w-converges to x.*

2. Characterizations of quasi H-closed spaces

In this section, we give some characterizations of quasi H-closed spaces and sets quasi H-closed relative to the space.

THEOREM 2.1. *For a topological space X, the following are equivalent*:

(i) *X is QHC.*

(ii) *Every regular open cover has a finite proximate subcover.*

(iii) *For each family of nonempty regular closed sets $\{F_\alpha\}$ such that $\cap F_\alpha = \phi$, there exists a finite subfamily $\{F_{\alpha_i}\}_{i=1}^n$ such that $\cap_{i=1}^n \mathrm{Int}(F_{\alpha_i}) = \phi$.*

(iv) *For each family of nonempty regular closed sets $\{F_\alpha\}$, if each finite subfamily $\{F_{\alpha_i}\}_{i=1}^n$ has the property that $\cap_{i=1}^n \mathrm{Int}(F_{\alpha_i}) \neq \phi$, then $\cap_\alpha F_\alpha \neq \phi$.*

(v) *Each filterbase $\mathcal{F} = \{A_i\}$ w-accumulates to somepoint $x \in X$.*

(vi) *Each maximum filterbase $\mathcal{F} = \{A_i\}$ w-converges.*

Proof. (i) $\Leftrightarrow$ (ii). It has been shown in [1] in view of the fact that for any open set U, Int (Cl (U)) is regular open.

(i) $\Rightarrow$ (vi). Let $\mathcal{F} = \{A_i\}$ be a maximal filterbase. Suppose that $\mathcal{F}$ does not w-converges to any point. Therfore, by Theorem 1.1, $\mathcal{F}$ does not w-accumulates to any point. This implies that for every $x \in X$, there exists an open set V_x containing x and $A_{ix} \in \mathcal{F}$ such that $A_{ix} \cap \mathrm{Cl}(V_x) = \phi$. Obviously, $\{V_x : x \in X\}$ is an open cover of X and by hypothesis, there exists a finite subfamily such that $X = \cup_{i=1}^n \mathrm{Cl}(V_{x_i})$. Since $\mathcal{F}$ is a filterbase, there exists an $A_k \in \mathcal{F}$ such that $A_k \subset \cap_{i=1}^n A_{ix_i}$. Hence $A_k \cap \mathrm{Cl}(V_{x_i}) = \phi$ for each $i = 1, 2, \dots, n$, which implies

$$A_k \cap (\cup_{i=1}^n \mathrm{Cl}(V_{x_i})) = A_k \cap X = \phi,$$

contradicting the essential fact that $A_k \neq \phi$. (vi) $\Rightarrow$ (v). Each filterbase is contained in a maximal filterbase.

(v) $\Rightarrow$ (iii). Let $\{F_\alpha\}$ be a family of regular closed sets such that $\cap_\alpha F_\alpha = \phi$. Suppose that for every finity subfamily, $\cap_{i=1}^n \mathrm{Int}(F_{\alpha_i}) \neq \phi$. Therefore $\mathcal{F} = \{\cap_{i=1}^n \mathrm{Int}(F_{\alpha_i}) : n \in N,\ F_{\alpha_i} \in \{F_\alpha\}\}$ forms a filterbase. By hypothesis, $\mathcal{F}$ w-accumulates to some $x \in X$. This implies that for every open set V_x containing x, $\mathrm{Int}(F_\alpha) \cap \mathrm{Cl}(V_x) \neq \phi$ for every $\alpha \in I$. Since $x \notin \cap F_\alpha$, there exists a $k \in I$ such that $x \notin F_k$. Hence x is contained in the regular open set $X - F_k$. Therefore $\mathrm{Int}(F_k) \cap \mathrm{Cl}(X - F_k) = \mathrm{Int}\ (F_k) \cap (X - \mathrm{Int}(F_k)) = \phi$, contradicting the fact that $\mathcal{F}$ w-accumulates to x.

(iii) $\Leftrightarrow$ (iv). Obvious.

(iii) $\Rightarrow$ (ii). Let $\{V_\alpha\}$ be a regular open cover of X. Then $\{X - V_\alpha\}$ is a family of regular closed sets satisfying $\cap_\alpha (X - V_\alpha) = \phi$. By hypothesis, there exists a finite subfamily such that $\cap_{i=1}^n \mathrm{Int}(X - V_{\alpha_i}) = \phi$.

Therefore $\cup_{i=1}^n \mathrm{Cl}(V_{\alpha_i}) = X$.

DEFINTION. A subset A of a space X is

(1) *RS-compact relative to* X (resp. *S-closed relative to* X [4]) *if for every cover* $\{V_\alpha : \alpha \in I\}$ *of* A *by regular semiopen* (resp. *semiopen*) *sets of* X, *there exists a finite subset* I_0 *of* I *such that* $A \subset \bigcup_{\alpha \in I_0} \mathrm{Int}(V_\alpha)$ (resp. $A \subset \bigcap_{\alpha \in I_0} \mathrm{Cl}(V_\alpha)$).

(2) *nearly compact relative to* X (resp. *quasi H-closed relative to* X [6]) *if for every cover* $\{V_\alpha : \alpha \in I\}$ *of* A *by open sets of* X, *there exists a finite subset* I_0 *of* I *such that* $A \subset \bigcup_{\alpha \in I_0} \mathrm{Int}(\mathrm{Cl}(V_\alpha))$ (resp. $A \subset \bigcup_{\alpha \in I_0} \mathrm{Cl}(V_\alpha)$).

LEMMA [4] *For a topological space, the following implications hold. If the space is extremally disconnected, then these four properties are equivalent*:

RS-compact $\Longrightarrow$ S-closed $\Longrightarrow$ quasi H-closed

$\qquad\qquad\qquad\qquad$ $\lfloor\!\Longrightarrow$ nearly compact $\longrightarrow \Uparrow$

An easy consequence of these is

THEOREM 2.2. *Let* X *be an extremally disconnected space. Then for a subset* A *of* X, *the following are equivalent*:

(i) A *is RS-compact relative to* X.

(ii) A *is S-closed relative to* X.

(iii) A *is nearly compact relative to* X.

(iv) A *is quasi H-closed relative to* X.

THEOREM 2.3. *A subset* A *of* X *is quasi H-closed relative to* X *if and only if for every cover* $\{V_\alpha : \alpha \in I\}$ *of* A *by regular open sets of* X, *there exists a finite subset* I_0 *of* I *such that* $A \subset \bigcup_{\alpha \in I_0} \mathrm{Cl}\,(V_\alpha)$.

Proof. The proof is similar to that of Theorem 2.1 and is thus omitted.

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