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# SOME PROPERTIES OF FUZZY TOPOLOGICAL SPACES

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## 0. Introduction

In his classical paper [22], Zadeh first introduced the fundamental concept of fuzzy sets. An immediate application of this idea can be found in the theory of general topology. Since fuzzy sets as introduced by Zadeh have the same kind of operations as set operation in general topology. It is, therefore, natural to extend the concept of point set topology to fuzzy sets, resulting in a theory of fuzzy topology. In the development of a parallel theory based on fuzzy sets, many interesting phenomena have been observed [1] [6] [11] [19-21]. Specially, one notices many differences between the two theories. A good example is the Tychonoff theorem in general topology: Any product of compact spaces is compact. Its fuzzy counterpart holds only for finite products. Since this approach to compactness in fuzzy spaces have serious limitations, various definitions of compactness in fuzzy spaces have been introduced.

Specially the authors in [5] proposed a new definition of compactness in fuzzy space. Also they defined the point-dependent Hausdorff separation axiom in fuzzy spaces. Rodabough [14] gave more general definition of Hausdorff separation axiom. Since many concepts and definitions in fuzzy topological spaces have not yet taken the their final forms, in this paper, we adopt Rodabough's approach to the theory of fuzzy topological spaces.

On the other hand, the systematic study of bitopolgical spaces [a set on which are defined two topologies] was begun by Kelly [8], who introduced various separation properties into bitopological spaces, and obtained generalizations of some important classical results. Also various other authors have contributed to the development of the theory([2] [4] [10] [12] [17]).

The purpose of this paper is to introduce the concept of fuzzy bitopological space, and to study some properties of fuzzy bitopological space.
In section II, we deal with some characterizations of pairwise $\alpha$-compact *fbs*.
In section III, we give the definition of some separation axiom of *fbs* and characterize some property of fbs satisfying the given separation axiom.
In section IV, we give the definition of pairwise $\alpha$-connectivity in a *fbs*

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and characterize some elementary property of pairwise $\alpha$-connected space.
In section V, we deal with the product theorem of a *fbs*.
For definitions and notations concerning bitopological and fuzzy topological spaces which are not explained in this paper, the reader is referred to [1–2], [4–6], [17], [19–21].

## I. Preliminary

In this section, we introduce some basic terminologies and definitions for the further study. We will let I denote the closed unit interval [0, 1] of the real line; in its natural order, I is a completely distributive lattice with order reversing involution defined by $a'=1-a[a\in I]$. Given an [ordinary] nonempty set $X$, the fuzzy sets of $X$ are just the elements of $I^X$, i.e., the functions from $X$ into $I$. The crisp subset of $X$ are just the $\{0, 1\}$-valued functions on $X$, i.e. the characteristic functions of the subsets of $X$ (We will identify a subset of $X$ with the associated crisp subset of $X$). If $a\in I$. then $a$ is the constant function from $X$ to $I$ whose value is $a$.

By a fuzzy topology on a set $X$ we mean a subset $\mathscr{T}\subset I^X$ such that (1) $\mathbf{0}$, $\mathbf{1}\in\mathscr{T}$. (2) $u$, $v\in\mathscr{T}\Rightarrow u\wedge v\in\mathscr{T}$ and (3) $\varphi\subset\mathscr{T}\Rightarrow\vee\varphi\in\mathscr{T}$. Here, $u\wedge v$ and are defined by $(u\wedge v)\ (x)=\inf\{u(x),\ v(x)\}$, $\vee\varphi(x)=\sup\{s(x):\ s\in\varphi\}$, for each $x\in X$. And $u\vee v$ and $\wedge\varphi$ are similarly defined. A base for a fuzzy topology $\mathscr{T}$ on a set $X$ is a collection $\mathscr{B}\subset\mathscr{T}$ such that, for each $u\in\mathscr{T}$ there exists $\mathscr{B}_u\subset\mathscr{B}$ with $u=\vee\mathscr{B}_u$; and a subbase for $\mathscr{T}$ is a collection $\varphi\subset\mathscr{T}$ such that $\{s_1\wedge\ldots\wedge s_n;n\in N$ and $s_1,\ \ldots,\ s_n\in\varphi\}\cup\{\mathbf{1}\}$ is a base for $\mathscr{T}$. Any collection $\varphi\subset I^X$ is a subbase for a unique fuzzy topology $T(\varphi)$ on $X$; we say that $T(\varphi)$ is generated by $\varphi$. By a fuzzy space we mean a pair $(X,\ \mathscr{T})$, where $X$ is a set and $\mathscr{T}$ is a fuzzy topology on $X$. The elements of $\mathscr{T}$ are called the open fuzzy sets of $X$, and their complements $u'$, where $u'(x)=1-u(x)$, are called the closed fuzzy sets of $X$. As in general topology, the indiscrete fuzzy topology contains only $\mathbf{0}$ and $\mathbf{1}$, while the discrete fuzzy topology contains all fuzzy sets. If $A$ is a subset of $X$, and if $\mathscr{T}$ is a fuzzy topology on $X$, then the the set of restrictions $\mathscr{T}_A=\{u|A:u\in\mathscr{T}\}$ is a fuzzy topology on $A$, and we say that $(A,\ \mathscr{T}_A)$ is a subspace of $(X,\ \mathscr{T})$. If $(X,\ \mathrm{P})$ and $(Y,\ Q)$ are two fuzzy spaces, then a map $f:X\to Y$ is $F$-continuous if $f^{-1}(v)\in\mathrm{P}$ for $v\in Q$, where $f^{-1}(v)(x)=v(f(x))$, for all $x\in X$.

The following definitions are useful in this paper.

1.1 DEFINITION [14] If $A\subset X$, $x\in X$ is an $\alpha$-cluster point of $A$ for $a$ $(\in[0,1)$ if for each $u\in\mathscr{T}$ such that $u(x)>\alpha$, $u\wedge\mu(A-\{x\})\neq\mathbf{0}$ where $\mu(A)$ is the characteristic function for $A$. The $\alpha$-closure of $A$ is the union of $A$ with the set of its $\alpha$-cluster points and is denoted by $\mathrm{Cl}_\alpha(A)$.

The set $A$ is $\alpha$-closed if $Cl_\alpha(A) \subset A$. Hence $A$ is $\alpha$-closed iff for each $x \in X-A$ there is $u \in \mathcal{T}$ such that $u(x) > \alpha$ and $u \wedge \mu(A) = \mathbf{0}$.

1.2 DEFINITION [14] $(X, \mathcal{T})$ has the $\alpha$-property if $\{A \subset X : A$ is $\alpha$-closed$\}$ $= [\{x \in X : u(x) \leq \alpha\},\ u \in \mathcal{T}]$. The left collection is always contained in the right collection. $(X, \mathcal{T})$ has the $\alpha$-property iff for each $x \in X$ and for each $u \in \mathcal{T}$ such that $u(x) > \alpha$, there is $v \in \mathcal{T}$ such that $v(x) > \alpha$ and $v(y) = 0$ for each $y \in X$ such that $u(y) \leq \alpha$.

1.3 DEFINITION [5] $(X, \mathcal{T})$ is $\alpha$-compact if each $\alpha$-shading of $X$ (a collection $\mathcal{U} \subset \mathcal{F}$ such that for each $x \in X$, there is $u \in \mathcal{U}$ such that $u(x) > \alpha$) has a finite $\alpha$-subshading of $X$.

For the remainder of this paper, we always assume $\alpha < 1$ when considering "$\alpha$-anything". Let $\{X_i\}$, $i \in \mathcal{A}$, be a family of sets. Let $X = \prod_{i \in \mathcal{A}} X_i$ be the usual product set, and let $p_i$ be the projection from $X$ onto $X_i$.

Further assume that each $X_i$ is a fuzzy topological space with fuzzy topology $\mathcal{T}_i$. Let $B \in \mathcal{T}_i$, $p_i^{-1}(B)$ is a fuzzy set in $X$. The family of fuzzy sets $S = \{p_i^{-1}(B) \mid B \in \mathcal{T}_i,\ i \in \mathcal{A}\}$ is now used to generate a fuzzy topology $\mathcal{T}$ for $X$ in the following manner. Let $\mathcal{B}$ be the family of all finite meets of members of $S$. Let $\mathcal{T}$ be the family of all joins of members of $\mathcal{B}$. It is clear that $\mathcal{T}$ is indeed a fuzzy topology for $X$, with $\mathcal{B}$ as a base and $S$ a subbase.

1.4 DEFINITION Given a family of fuzzy topological spaces $\{(X_i, \mathcal{T}_i),\ i \in \mathcal{A}\}$, the fuzzy topology defined as above is called the product fuzzy topology for $X = \prod_{i \in \mathcal{A}} X_i$ and $(X, \mathcal{T})$ is called the product fuzzy topologiacl space.

1.5 DEFINITION A space $X$ on which are defined two (arbitrary) fuzzy topologies $P$ and $Q$ is called a fuzzy bitopological space, or *fbs* for short, and denoted by $(X, P, Q)$.

1.6 DEFINITION A function $f$ mapping a *fbs* $(X, P, Q)$ into a *fbs* $(X^*, P^*, Q^*)$ will be said to be pairwise $F$-continuous iff the induced mappings $f : (X, P) \to (X^*, P^*)$ and $f : (X, Q) \to (X^*, Q^*)$ of the fuzzy topological spaces are $F$-continuous.

## II. Pairwise $\alpha$-compactness

An $\alpha$-shading $\mathcal{U}$ of the fbs $(X, P, Q)$ is a collection $\mathcal{U} \subset P \cup Q$ such that for each $x \in X$, there is $u \in \mathcal{U}$ such that $u(x) > \alpha$.

2.1. DEFINITION An $\alpha$-shading $\mathcal{U}$ of the *fbs* $(X, P, Q)$ is called pairwise $\alpha$-shading if $\mathcal{U}$ contains at least one member $u(\{x : u(x) > \alpha\} \neq \phi)$ of $P$ and at least one member $v(\{x : v(x) > \alpha\} \neq \phi$ of $Q$

2.2 DEFINITION A *fbs* $(X, P, Q)$ is called pairwise $\alpha$-compact if every pairwise $\alpha$-shading of $(X, P, Q)$ has a finite $\alpha$-subshading.

2.3 DEFINITION A *fbs* $(X, P, Q)$ has the $\alpha$-property if both $(X, P)$ and $(X, Q)$ have the $\alpha$-property.

To obtain two characterizations of pairwise $\alpha$-compactness, we need the notion of adjoint fuzzy topology similar to adjoint topology which was introduced by Kim[10]. If $\mathcal{T}$ is a fuzzy topology on $X$ and $a$ is a fuzzy set (not $\mathbf{0}$) of $X$, then $\mathcal{T}(a)$ is the fuzzy topology on $X$ given by $\mathcal{T}(a)=\{\mathbf{0}, \mathbf{1}\} \cup \{a \vee u : u \in \mathcal{T}\}$.

2.1 THEOREM *If $(X, P, Q)$ is a fbs, then the conditions below are related as follows. (a) implies (b), and (b) implies (c). If $(X, P, Q)$ has the $\alpha$-property, then (a), (b) and (c) are equivalent.*

(a) *$(X, P, Q)$ is pairwise $\alpha$-compact.*

(b) *For each fuzzy set (not $\mathbf{0}$) $a$ in $P$, the fuzzy space $(X, Q(a))$ is $\alpha$-compact and for each fuzzy set (not $\mathbf{0}$) $a$ in $Q$, the fuzzy space $(X, P\ (a))$ is $\alpha$-compact.*

(c) *Each $P$-$\alpha$-closed proper subset is $Q$-$\alpha$-compact and each $Q$-$\alpha$-closed proper subset is $P$-$\alpha$-compact.*
*(Where $P$-$\alpha$-closed set, $Q$-$\alpha$-compact set denote the $\alpha$-closed set in $(X, P)$ and $\alpha$-compact set in $(X, Q)$ respectively)*

*Proof.* (a) $\Rightarrow$ (b). Let a be any open fuzzy set (not $\mathbf{0}$) in $(X. P)$, and $\mathcal{U}$ be an $\alpha$-shading of $(X, Q(a))$, so $\mathcal{U}=\{a \vee u_i : i \in \mathcal{A}\}$ where $u_i \in Q$ for each $i \in \mathcal{A}$. Then the collection $\{a\} \cup \{u_i : i \in \mathcal{A}\}$ is a pairwise $\alpha$-shading of $(X, P, Q)$ and so has a finite $\alpha$-subshading which we denote by $\{a\} \cup \{u_i\ ;\ i=1, 2, \dots. n\}$. We add $\{a\}$ to the $\alpha$-subshading if necessary. Then $\{a \vee u_i : i=1, 2, \dots, n\}$ is the desired finite $\alpha$-subhading of $\mathcal{U}$ for $(X, Q(a))$, so that the fuzzy space $(X, Q(a))$ is $\alpha$-compact. Similarily, the fuzzy space $(X, P(a))$ is $\alpha$-compact for each open fuzzy set $a$ (not $\mathbf{0}$) in $(X, Q)$.

(b) $\Rightarrow$ (c). Let $K$ be any proper $P$-$\alpha$-closed subset. Since $K$ is $\alpha$-closed, for each $x \in X-K$, there is $u_x \in P$ such that $u_x(x) > \alpha$ and $u_x \wedge \mu(K)=\mathbf{0}$. Put $u=\bigvee_{x \in X-K} u_x$, then $u \in P$ and $u(x) > \alpha$ for any $x \in X-K$. Let $\{u_i : i \in \mathcal{A}\}$ be an $\alpha$-shading of $K$ in $(X, Q)$. Then the collection $\{u \vee u_i : i \in \mathcal{A}\}$ is an $\alpha$-shading of $(X, Q(u))$, so the collection $\{u \vee u_i : i \in \mathcal{A}\}$ has a finite $\alpha$-subshading $\{u \vee u_i : i=1, \dots, n\}$ of $(X, Q(u))$. Since $u \wedge \mu(K)=\mathbf{0}$, the collection $\{u_i : i=1, \dots, n\}$ is a finite $\alpha$-subshading of $K$, so that $K$ is $\alpha$-compact in $(X, Q)$. Similarily, each $Q$-$\alpha$-closed proper subset is $\alpha$-compact in $(X, P)$.
If $(X, P, Q)$ has the $\alpha$-property (c) $\Rightarrow$ (a). Let $\mathcal{U}$ be a pairwise $\alpha$-shading of $(X, P, Q)$, let the $P$ open fuzzy sets in $\mathcal{U}$ be $\{u_i : i \in \mathcal{B}\}$, and let the $Q$

open fuzzy set in $\mathcal{U}$ be $\{v_j : j\in\mathcal{A}\}$. Two cases arise.
(i) $\{v_j : j\in\mathcal{A}\}$ is $\alpha$-shading of $(X, Q)$. Choose a $i_0\in\mathcal{B}$ such that $u_{i_0}(\{x : u_{i_0}(x)>\alpha\} \neq \phi)$ since $(X, P, Q)$ has the $\alpha$-property, the set $\{x : u_{i_0}(x)\leq\alpha\}$ is $P$-$\alpha$-closed proper subset. Then the collection $\{v_j : j\in\mathcal{A}\}$ is an $\alpha$-shading of the set $\{x : u_{i0}(x)\leq\alpha\}$, so there is a finite $\alpha$-subshading $\{v_j : j=1, ..., m\}$ of the set $\{x : u_{i0}(x)\leq\alpha\}$. Then $\{u_{i0}, v_1, ..., v_m\}$ is a finite $\alpha$-subshading of $\mathcal{U}$ for $(X, P, Q)$.
(ii) $\{v_j : j\in\mathcal{A}\}$ is not $\alpha$-shading of $(X, Q)$. Then the set $K=\{x : \bigwedge_{j\in\mathcal{A}} v_j(x)\leq \alpha\}$ is a $Q$-$\alpha$-closed proper subset and $\{u_i : i\in\mathcal{B}\}$ is an $\alpha$-shading of K. Hence there is a finite $\alpha$-subshading $\{u_i : i=1. ..., n\}$ of $K$. If $\{u_i : i=1, ..., n\}$ is an $\alpha$-shading of $(X, P)$, there is nothing more to prove. If $\{u_i : i=1, ..., n\}$ is not an $\alpha$-shading of $(X, P)$, then $\{x : \bigvee_{i=1}^{n} u_i(x)\leq\alpha\}$ is a p-$\alpha$-closed proper subset and $\{v_j : j\in\mathcal{A}\}$ is an $\alpha$-shading of $\{x : \bigvee_{i=1}^{n} u_i(x)\leq\alpha\}$. By hypothesis, there is a finite $\alpha$-shading $\{v_i : i=1, ..., p\}$ of $\{x : \bigvee_{i=1}^{n} u_i(x)\leq\alpha\}$. Then $\{u_i : i=1, ..., n\} \cup \{v_j : j=1, ..., p\}$ is the required finite $\alpha$-subshading of $\mathcal{U}$ for $(X, P, Q)$.

As is shown by the following example, the hypothesis that $(X, P, Q)$ has the $\alpha$-property is non-superfluous in the above theorem.

EXAMPLE In this example we assume $\alpha>0$. Let $X=\{x_1, x_2, x_3, ...\}$ be a countable set. Let $u_1 : X\to I$ such that $u_1(x_1)>\alpha$, $u_1(x_i)=\frac{\alpha}{2}$, and for $j\geq 2 u_j : X\to I$ such that $u_j(x_1)=0$, $u_j(x_j)>\alpha$, $u_j(x_k)=\frac{\alpha}{2}$ when $j\neq k$, Let $v_2 : X\to I$ such that $v_2(x_2)>\alpha$, $v_2(x_i)=\frac{\alpha}{2}$ and for $j\neq 2$ $v_j : X\to I$ such that $v_j(x_2)=0$, $v_j(x_k)=\frac{\alpha}{2}$, $v_j(x_j)>\alpha$. Let $P$ and $Q$ be the fuzzy topologies generated by $\{u_1, u_2, ...\}$ and $\{v_1, v_2, ...\}$, then it is clear that the *fbs* $(X, P, Q)$ has not the $\alpha$-property.

Also we can easily see that $\{x_1\}$ and $\{x_2\}$ are the only $\alpha$-closed non-empty proper subset in $(X, P)$ and $(X, Q)$. Hence $\{x_1\}$ is $\alpha$-compact in $(X, Q)$ and $\{x_2\}$ is $\alpha$-compact in $(X, P)$. But the pairwise $\alpha$-shading $\{u_1, v_2, u_3, v_4, ...\}$ for $(X, P, Q)$ has not a finite $\alpha$-subshading.

2. 2 THEOREM *The pairwise F-continuous image of pairwise $\alpha$-compact fbs is pairwise $\alpha$-compact.*

*Proof.* Suppose $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise $\alpha$-compact, let $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (X^*, \mathcal{T}_1^*, \mathcal{T}_2^*)$ be a pairwise $F$-continuous and let $\{u_i\}$ be a pairwise $\alpha$-shading of $f(X)$ in $(X^*, \mathcal{T}_1^*, \mathcal{T}_2^*)$.

Then $f^{-1}(u_i)$ is a pairwise $\alpha$-shading of $(X, \mathscr{T}_1, \mathscr{T}_2)$ and so by the pairwise $\alpha$-compactness of $(X, \mathscr{T}_1, \mathscr{T}_2)$. It follows that there is a finite $\alpha$-subshading of $(X, \mathscr{T}_1, \mathscr{T}_2)$, say $f^{-1}(u_1), \ldots, f^{-1}(u_n)$. The corresponding family $\{u_1, u_2, \ldots, u_n\}$ then forms the required finite $\alpha$-subshading of the $\{u_i\}$.

## III. Separation axioms of fuzzy bitopological spaces

In this section, we give the definition of some separation axiom of fbs and characterize some property of fbs satisfying the given separation axiom.

3. 1. DEFINITION The *fbs* $(X, P, Q)$ is called pairwise $\alpha$-$T_1$ if for every pair of distinct points $x, y \in X$, there is $u \in P$ or $v \in Q$ such that $u(x) > \alpha$, $u(y) = 0$ or $v(x) > \alpha$, $v(y) = 0$.

3. 2. DEFINITION The *fbs* $(X, P, Q)$ is called pairwise $\alpha$-Hausdorff if for every pair of distinct points $x, y \in X$, there is $u \in P$, $v \in Q$ such that $u(x) > \alpha$, $v(y) > \alpha$, and $u \wedge v = \mathbf{0}$ and there is $u' \in Q$, $v' \in P$ such that $u'(x) > \alpha$, $v'(*y) > \alpha$, $u' \wedge v' = \mathbf{0}$.

3. 3. DEFINITION The *fbs* $(X, P, Q)$ is called pairwise $\alpha$-regular if it sati sfies the following:

i) for each point $x \in X$ and each $P$-$\alpha$-closed set $A$ such that $x \notin A$, there is $u \in P$, $v \in Q$ such that $u(x) > \alpha$. $v(A) > \alpha$, and $u \wedge v = \mathbf{0}$.

ii) for each point $x \in X$ and each $Q$-$\alpha$-closed set $A$ such that $x \notin A$, there is $u' \in P$, $v' \in Q$ such that $v'(x) > \alpha$, $u'(A) > \alpha$, and $u' \wedge v' = \mathbf{0}$.

3. 4. DEFINITION The *fbs* $(X, P, Q)$ is called pairwise $\alpha$-normal if given a $P$-$\alpha$-closed set $A$ and a $Q$-$\alpha$-closed set $B$ such that $A \cap B = \phi$, there is $u \in P, v \in Q$ such that $u(B) > \alpha$, $v(A) > \alpha$ and $u \wedge v = \mathbf{0}$.

3. 1. THEOREM *The following statements are equivalent.*

(a) $(X, P, Q)$ *is pairwise* $\alpha$-$T_1$.

(b) $P$-$\mathrm{Cl}_\alpha\{x\} \cap Q$-$\mathrm{Cl}_\alpha\{x\} = \{x\}$ *for each* $x \in X$.

(c) $[\cap \{x \mid u(z) > 0\} : u(x) > \alpha, u \in P\} \cap [\cap \{z \mid v(z) > 0\} : v(x) > \alpha, v \in Q\}] = \{x\}$ *for each* $x \in X$.

*Proof.* (a) $\Rightarrow$ (*b*). For any $y \in X$ such that $x \neq y$, from the assumption, there is $u \in P$ or $v \in Q$ such that $u(y) > \alpha$, $u(x) = 0$ or $v(y) > \alpha$, $v(x) = 0$. Hence we have $y \notin P$-$\mathrm{Cl}_\alpha\{x\}$ or $y \notin Q$-$\mathrm{Cl}_\alpha\{x\}$. Thus $y \notin P$-$\mathrm{Cl}_\alpha\{x\} \cap Q$-$\mathrm{Cl}_\alpha\{x\}$.

(b) $\Rightarrow$ (c). If $y \neq x, y \in [\cap \{x \mid u(z) > 0\} : u(x) > \alpha, u \in P] \cap [\cap \{z \mid v(z) > 0\} : v(x) > \alpha, v \in Q]$ then $y \in [\cap \{z \mid u(z) > 0\} : u(x) > \alpha, u \in P]$ and $y \in [\cap \{z \mid v(z) > 0\} : v(x) > \alpha, v \in Q]$. From the definition of $\alpha$-closure $x \in P$-$\mathrm{Cl}_\alpha\{y\}$ and $x \in Q$-$\mathrm{Cl}_\alpha\{y\}$, hence $x \in P$-$\mathrm{Cl}_\alpha\{y\} \cap Q$-$\mathrm{Cl}_\alpha\{y\}$. This is a contradiction.

(c) $\Rightarrow$ (a). It is clear from the definition of $\alpha$-cluster point.

3.2. THEOREM *The following statements are equivalent.*

(a) $(X, P, Q)$ *is pairwise $\alpha$-Hausdorff.*

(b) *For $x, y \in X$ such that $x \neq y$, there is $u \in P$ such that $u(x) > \alpha$, $u(y) = 0$ and $y \notin Q\text{-Cl}_\alpha \{z : u(z) > 0\}$ : also there is $v \in Q$ such that $v(y) > \alpha$, $v(x) = 0$ and $x \notin P\text{-Cl}_\alpha \{z : v(z) > 0\}$.*

(c) *For each $x \in X$, $\cap \{Q\text{-Cl}_\alpha \{z | u(z) > 0\} : u(x) > \alpha, u \in P\} = \{x\} = \cap \{P\text{-Cl}_\alpha \{z | v(z) > 0\} : v(x) > \alpha, v \in Q\}$.*

(d) *The diagonal $\Delta = \{(x, x) : x \in X\}$ is $\alpha$-closed in $(X \times X, P \times Q)$ and $(X \times X, Q \times P)$.*

*Proof.* (a)$\Rightarrow$(b). For $x, y \in X$ such that $x \neq y$, there is $u \in P$, $v \in Q$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \wedge v = \mathbf{0}$.
$v(y) > \alpha$ implies $u(y) = 0$. Also $u \wedge v = \mathbf{0}$ implies $v(z) = 0$ for $z \in \{z : u(z) > 0\}$ hence $v(y) > \alpha$ implies y is not $Q$-$\alpha$-cluster point of the set $\{z : u(z) > 0\}$.

Similarily there is $v \in Q$ such that $v(y) > \alpha$, $v(x) = 0$ and $x \notin P\text{--Cl}_\alpha \{z : v(z) > 0\}$.

(b)$\Rightarrow$(c). It is clear.

(c)$\Rightarrow$(d). Let $(x, y) \in X \times X - \Delta$, we have $u \in P$, $v \in Q$ such that $u(x) > \alpha$, $v(x) > \alpha$, and $y \notin Q\text{-Cl}_\alpha \{z | u(z) > 0\}$ $y \notin P\text{—Cl}_\alpha \{z | v(z) > 0\}$, and hence we have $v' \in P$ such that $v'(y) > \alpha$ and $v' \wedge \mu \{z : v(z) > 0\} = \mathbf{0}$. Let $f = p_1^{-1}(v) \wedge p_2^{-1}(v')$ where $p_1$ and $p_2$ are projections, then we have $f(x, y) > \alpha$ and $f \wedge \mu(\Delta) = \mathbf{0}$, therefore $\Delta$ is $\alpha$-closed in $(X \times X, Q \times P)$.

(d)$\Rightarrow$(a). Let $x, y \in X$ such that $x \neq y$. Since $\Delta$ is $\alpha$-closed in $(X \times X, P \times Q)$ and $(X \times X, Q \times P)$, there is an open fuzzy set $u_\Delta$ in $(X \times X, P \times Q)$ such that $u_\Delta(x, y) > \alpha$ and $u_\Delta \wedge \mu(\Delta) = \mathbf{0}$. Hence there is $u \in P$, $v \in Q$ such that $u_\Delta \geq p_1^{-1}(u) \wedge p_2^{-1}(v)$ and $[p_1^{-1}(u) \wedge p_2^{-1}(v)](x, y) > \alpha$. Then for each $(z, z) \in \Delta u_\Delta(z, z) \geq u(z) \wedge v(z)$ implies $u \wedge v = \mathbf{0}$ and we have that $u(x) > \alpha$ and $v(y) > \alpha$.

In an ordinary topology, it is well known that a compact set in a Hausdorff space is closed. In a *fbs*, we have the similar results.

3.3. THEOREM *Suppose a fbs $(X, P, Q)$ is pairwise $\alpha$-Hausdorff, then every $P$-$\alpha$-compact set is $Q$-$\alpha$-closed and every $Q$-$\alpha$-compact set is $P$-$\alpha$-closed.*

*Proof.* Let $A$ be a $P$-$\alpha$-compact set and let $p \in X - A$. Since $(X, P, Q)$ is pairwise $\alpha$-Hausdorff, for each $x \in A$, there is $u_x \in P$, $v_x \in Q$ such that $u_x(x) > \alpha$, $v_x(p) > \alpha$ and $u_x \wedge v_x = \mathbf{0}$. Since $\mathcal{U} = \{u_x | x \in A\}$ is an $\alpha$-shading of $A$, there is a finite $\alpha$-subshading $\{u_{x1}, u_{x2}, \ldots, u_{xn}\}$ of $\mathcal{U}$. If $u = \bigvee_{i=1}^{n} u_{xi}$ and $v = \bigwedge_{i=1}^{n} v_{xi}$ then $v \in Q$. Since $u \wedge v = \mathbf{0}$. $v(p) >$ and$v(A) = 0$. Thus $A$ is $Q$-$\alpha$-closed.

Similarily every $Q$-$\alpha$-compact set is $P$-$\alpha$-closed.

3.4. THEOREM *If a fbs $(X, P, Q)$ is pairwise $\alpha$-compact and pairwise $\alpha$-Hausdorff then $(X, P, Q)$ has the $\alpha$-property.*

*Proof.* It suffices to show that $(X, P)$ has the $\alpha$-property, that is, for any $u\in P$ the set $A=\{x : u(x)\leq\alpha\}$ is $P$-$\alpha$-closed. Let $p\in X$-$A$. Since $(X, P, Q)$ is pairwise $\alpha$-Hausdorff, for any $x\in A$, there is $u_x\in P$, $v_x\in Q$ such that $u_x(p)>\alpha$, $v_x(x)>\alpha$ and $u_x\wedge v_x=\mathbf{0}$. Since $\mathcal{U}=\{v_x : x\in A\}\cup\{u\}$ is a pairwise $\alpha$-shading of $(X, P, Q)$, there is a finite $\alpha$-subshading $\{v_{x1}, \dots, v_{xn}\}\cup\{u\}$ of $\mathcal{U}$. If $w=\bigwedge_{i=1}^{n} u_{x_i}$ and $v=\bigvee_{i=1}^{n} v_{x_i}$ then $w\in P$, $w(p)>\alpha$ and $w(A)=\mathbf{0}$. Thus $A$ is $P$-$\alpha$-closed.

The converse of the above theorem is not true, as is shown by the following example.

EXAMPLE Let $P$ and $Q$ be the discrete fuzzy space and the indiscrete fuzzy space on a set $X$. Then it is clear that a *fbs* $(X, P, Q)$ has the $\alpha$-property since $(X, P)$ and $(X, Q)$ have the $\alpha$-property. But $(X, P, Q)$ is not pairwise $\alpha$-Hausdorff.

3.5. THEOREM *If a fbs $(X, P, Q)$ is pairwise $\alpha$-Hausdorff and pairwise $\alpha$-compact then $(X, P, Q)$ is pairwise $\alpha$-regular.*

*Proof.* Let $C$ be a $P$-$\alpha$-closed subset and let $p\notin C$. Since $(X, P, Q)$ is pairwise $\alpha$-Hausdorff, for each $x\in C$ there is $u_x\in P$, $v_x\in Q$ such that $u_x(p)>\alpha$, $v_x(x)>\alpha$ and $u_x\wedge v_x=\mathbf{0}$. Now $\mathcal{U}=\{v_x | x\in C\}$ is a $Q$-$\alpha$-shading of $C$. Thus, by theorem 2.1, there is a finite $\alpha$-subshading $\{v_{x1}, \dots, v_{xn}\}$ of $\mathcal{U}$. If $u=\bigwedge_{i=1}^{n} u_{xi}$ and $v=\bigvee_{i=1}^{n} v_{xi}$ then $u\in P$, $v\in Q$, $u(p)>\alpha$, $v(C)>\alpha$ and $u\wedge v=\mathbf{0}$.

3.6. THEOREM *If a fbs $(X, P, Q)$ is pairwise $\alpha$-Hausdorff and pairwise compact then $(X, P, Q)$ is pairwise $\alpha$-normal.*

*Proof.* Let $H$ be a $P$-$\alpha$-closed set and $K$ be a $Q$-$\alpha$-cloeased set such that $H\cap K=\phi$. For each $x\in K$, there is $u_x\in P$, $v_x\in Q$ such that $u_x(x)>\alpha$, $v_x(H)>\alpha$ and $u_x\wedge v_x=\mathbf{0}$.
Since $\mathcal{U}=\{u_x | x\in K\}$ is an $\alpha$-shading of $K$, by theorem 2.1, there is a finite $\alpha$-subshading $\{u_{x1}, \dots, u_{xn}\}$ of $\mathcal{U}$. If $u=\bigvee_{i=1}^{n} u_{xi}$ and $v=\bigwedge_{i=1}^{n} v_{xi}$ then $u\in P$, $v\in Q$, $u(K)>\alpha$, $v(H)>\alpha$ and $u\wedge v=\mathbf{0}$.

## IV. Pairwise $\alpha$-connectivity

Rodabough (15) introduced $\alpha$-connectivity in a fuzzy space, slmilarily we define pairwise $\alpha$-connectivity in a *fbs* and characterize some elementary property of pairwise $\alpha$-connected space.

4. 1. DEFINITION A *fbs* $(X, P, Q)$ is called pairwise $\alpha$-connected if there is not $u \in P$, $v \in Q$ and $u$, $v \notin \{\mathbf{0}, \mathbf{1}\}$ such that on $X$, $u \vee v > \alpha$ and $u \wedge v = \mathbf{0}$. If $B \subset X$ and $(X, P, Q)$ is a fbs, then $B$ is pairwise $\alpha$-connected if $B$ is pairwise $\alpha$-connected in the fuzzy subspace $(B, P/B, Q/B)$.

4. 1. THEOREM *Let* $(X, P, Q)$ *be a* fbs. *Then the following statements hold.*

(a) *Countable unions of pairwise intersecting pairwise $\alpha$-connected sets are pairwise $\alpha$-connected.*

(b) $(X, P, Q)$ *is pairwise $\alpha$-connected iff there is not a non-empty proper subset $A$ of $X$ such that $A$ is $P$-$\alpha$-closed and $X$-$A$ is $Q$-$\alpha$-closed respectively.*

(c) *Suppose $A$ is pairwise $\alpha$-connected. Then $B$ is pairwise $\alpha$-connected if* $A \subset B \subset P\text{-}\mathrm{Cl}_\alpha(A) \cap Q\text{-}\mathrm{Cl}_\alpha(A)$.

*Proof.* (a) It suffices to show that $C = A \cup B$ is pairwise $\alpha$-connected if A and B are pairwise $\alpha$-connected and $A \cap B \neq \phi$.
Suppose $C$ is not pairwise $\alpha$-connected, then there is $u \in P/C$, $v \in Q/C$, and $u$, $v \notin \{\mathbf{0}/C, \mathbf{1}/C\}$ such that $u \vee v > \alpha$ and $u \wedge v = \mathbf{0}$ on $C$. It follows by case work that either each of $u/A$, $v/A$ is not in $\{\mathbf{0}/A, \mathbf{1}/A\}$ or each of $u/B$, $v/B$ is not in $\{\mathbf{0}/B, \mathbf{1}/B\}$. If the latter holds, $(u/B \vee v/B) > \alpha$ and $u/B \wedge v/B = \mathbf{0}$. This is a contradiction. Hence $C$ is pairwise $\alpha$-connected.

(b) Let $(X, P, Q)$ be pairwise $\alpha$-connected. If there is a non-empty proper subset $A$ of $X$ such that $A$ is $P$-$\alpha$-closed and $X$-$A$ is $Q$-$\alpha$-closed respectively, then that $A$ and $X$-$A$ are $P$-$\alpha$-closed and $Q$-$\alpha$-closed respectively implies that for each $x \in A$, $y \in X\text{-}A$ there is $u_y \in P$, $v_x \in Q$ such that $u_y(y) > \alpha$, $u_y/A = \mathbf{0}$ and $v_x(x) > \alpha$, $v_x/X\text{-}A = \mathbf{0}$. Let $u = \bigvee_{y \in X-A} u_y$, $v = \bigvee_{x \in A} v_x$. It follows that $u \in P$, $v \in Q$, $u \vee v > \alpha$ and $u \wedge v = \mathbf{0}$ on $X$. This is a contradiction.

Conversely, if $(X, P, Q)$ is not pairwise $\alpha$-connected, then there is $u \in P$, $v \in Q$ such that $u \vee v > \alpha$, $u \wedge v = \mathbf{0}$ and $u, v \notin \{\mathbf{0}, \mathbf{1}\}$. So $A = \{x : u(x) > \alpha\}$ and $B = \{x : v(x) > \alpha\}$ are non-empty proper $X$-$\alpha$-closed and $P$-$\alpha$-closed subsets such that $B = X\text{-}A$ respectively. This is a contradiction.

(c) To show this, we need only consider $A \neq \phi$, $A \subsetneq B$. If $B$ is not pairwise $\alpha$-connected, it follows there is $u \in P, v \in X$ such that either $A \subset \{x : u(x) > \alpha\}$ and $(B\text{-}A) \cap \{x : v(x) > \alpha\} \neq \phi$ or $A \subset \{x : v(x) > \alpha\}$ and $(B\text{-}A) \cap \{x : u(x) > \alpha \neq \phi$. This contradicts the assumption that $B \subset P\text{-}\mathrm{Cl}_\alpha(A) \cap Q\text{-}\mathrm{Cl}_\alpha(A)$.

It is clear from the above theorem that a pairwise $\alpha$-connected subset is contained in a maximal pairwise $\alpha$-connected subsets which we shall call the pairwise $\alpha$-component of the space. The next theorem is a generalization of the fact that the components of an ordinary topological space are closed.

4. 2. THEOREM *Any pairwise $\alpha$-component C o fa fbs $(X, P, Q)$ satisfies the equation* $C=P\text{-Cl}_\alpha(C) \cup Q\text{-Cl}_\alpha\ (C)$.

*Proof.* Let $C$ be a pairwise $\alpha$-component and suppose that $p \notin C$, then $C \cup \{p\}$ is not pairwise $\alpha$-connected. Hence there is $u \in P$, $v \in Q$ such that $u \vee v > \alpha$, $u \wedge v = \mathbf{0}$ on $C \cup \{p\}$ and $u/C \cup \{p\}$, $v/C \cup \{p\} \notin \{\mathbf{0}/C \cup \{p\}, \mathbf{1}/C \cup \{p\}\}$ Thus either $C \subset \{x : u(x) > \alpha\}$ and $v(p) > \alpha$ or $C \subset \{x : v(x) \supset \alpha\}$ and $u(p) > \alpha$. Hence either $p \notin P\text{-Cl}_\alpha(C)$ or $p \notin Q\text{-Cl}_\alpha(C)$.
This is equivalent to saying that $p \notin P\text{-Cl}_\alpha(C) \cap Q\text{-Cl}_\alpha(C)$ and so we have $P\text{-Cl}_\alpha(C) \cap Q\text{-Cl}_\alpha(C) \subset C$. Clearly $C \subset P\text{-Cl}_\alpha(C) \cap Q\text{-Cl}_\alpha(C)$, and the equation is satisfied.

4. 3. THEOREM *pairwise F-continuity preserves pairwise $\alpha$-connectivity.*

*Proof.* The assertion follows from the fact that if $f : X \to Y$ and $u$ is a fuzzy set in $Y$ then $f^{-1}(u)\ (x) = u(f(x))$.

## V. Product of fuzzy bitopological spaces

Let $(X_i, P_i, Q_i)$ be any family of fuzzy topological spaces. We construct in a natural way two fuzzy topologies on the cartesian product $\prod X_i$. Let $P$ be the fuzzy product topology on $\prod X_i$ determined by the $p_i$'s, that is, having as a subbase the family of all fuzzy sets of the form $p_i^{-1}(G)$ where $i$ is any index and $G \in P_i$. Let $Q$ be the fuzzy product topology on $\prod X_i$ determined by the $Q_i$'s. The resulting fbs $(\prod X_i,\ P,\ Q)$ will be called the fuzzy product bitopological space generated by the family $\{(X_i, P_i, Q_i)\}$.

The following results are presented without proof, as they are immediate consequence of the definition.

5. 1. THEOREM *Let $(X_i, P_i, Q_i)$ be an arbitrary family of fuzzy bitopological spaces. Then for each fixed k, the projection map $p_k : (\prod X_i, P, Q) \to (X_k, P_k, Q_k)$ is a pairwise F-continuous surjection.*

5. 2. THEOREM *Let $\{(X_i, P_i, Q_i)\}$ be any family of fuzzy bitopological spaces and let $f : (Y, \mathfrak{T}_1, \mathfrak{T}_2) \to (\prod X_i, P, Q)$ be any map. Then f is pairwise F-continuous iff $p_k \circ f$ is pairwise F-continuous for each k.*

The following three theorems are generalizations of known classical results.

5. 3. THEOREM *Let $\{X_i, \mathrm{p}_i, Q_i)\}$ be a family of non-empty fuzzy bitopological spaces. Then $(\prod X_i, P, Q)$ is pairwise $\alpha$-Hausdorff iff $(X_i, P_i, Q_i)$ is pairwise $\alpha$-Hausdorff for each i.*

*Proof.* Suppose each $(X_i, P_i, Q_i)$ is pairwise $\alpha$-Hausdorff. Let $\{x_i\}$ and $\{y_i\}$ be two distinct point in $X_i$. Then there is $k$ such that $x_k \neq y_k$. Since $(X_k, P_k, Q_k)$ is pairwise $\alpha$-Hausdorff, there is $u \in P_k$, $v \in Q_k$ such that $u(x_k) > \alpha$, $v(y_k) > \alpha$ and $u \wedge v = \mathbf{0}$. Then $p_k^{-1}(u)$ $(\{x_i\}) > \alpha$, $p_k^{-1}(v)$ $(\{y_i\}) > \alpha$ and $p_k^{-1}(u) \wedge p_k^{-1}(v) = \mathbf{0}$. Hence $(\pi X_i, P, Q)$ is pairwise $\alpha$-Hausdorff. Conversely, suppose $(\pi X_i, P, Q)$ is pairwise $\alpha$-Hausdorff. Let $x_k, y_k \in X_k$ such that $x_k \neq y_k$ and let $X = X_k x \{x_i : i \neq k\}$. Then $x = x_k x \{x_i : i \neq k\} \neq y_k x \{x_i : i \neq k\} = y$. Since a subspace of a pairwise $\alpha$-Hausdorff *fbs* is again pairwise $\alpha$-Hausdorff. Now $(X, P/X, Q/X)$ is pairwise $\alpha$-Hausdorff implies that there is $u \in P/X$, $v \in Q/X$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \wedge v = \mathbf{0}$. Note that $u$ and $v$ are each of the form $\vee \mathcal{A} \{[p_{\gamma 1}^{-1}(u_{\gamma 1}) \wedge \ldots \wedge p_{\gamma n}^{-1}(u_{\gamma n})] : u_{\gamma i} \in P_{\gamma i}, \gamma_1, \gamma_2, \ldots \gamma_n \in \mathcal{A}\}$ and $\vee \mathcal{B} \{[p_{\delta 1}^{-1}$ $(v_{\delta 1}) \wedge \ldots \wedge p_{\delta m}^{-1}$ $(v_{\delta m})] : v_{\delta i} \in Q_{\delta i}, \delta_1, \delta_2 \ldots \delta_m \in \mathcal{B}\}$ and note if any $\gamma_i \neq k$ and $\delta_i \neq k$, then $p_{\gamma i}^{-1}(u_{\gamma i})$ and $p_{\delta i}^{-1}$ $(v_{\delta i})$ are constant on $X$. thus $u \geq p_k^{-1}$ $(u_k) \wedge k_1$, $v \geq p_k^{-1}(v_k) \wedge k_2$. where $u_k \in P_k$, $v_k \in Q_k$, $[p_k^{-1}(u_k) \wedge k_1](x) > \alpha$ and $[p_k^{-1}(v_k) \wedge k_2](y) > \alpha$. Since $k_1 \wedge k_2 > 0$, $0 = u \wedge v \geq p_k^{-1}(u_k \wedge v_k) \wedge (k_1 \wedge k_2)$ implies $p_k^{-1}(u_k \wedge v_k) = \mathbf{0}$, so $u_k \wedge v_k = \mathbf{0}$. Also $p_k^{-1}(u_k)(x) > \alpha$ and $p_k^{-1}(v_k)(y) > \alpha$ so $u_k(x_k) > \alpha$ and $v_k(y_k) > \alpha$. Hence $(X_k, P_k, Q_k)$ is pairwise $\alpha$-Hausdorff space.

5.4. THEOREM *If $(X_i, P_i, Q_i)$ is a family of fuzzy bitopological spaces such that $(\pi X_i, P, Q)$ is pairwise $\alpha$-compact, then each $(X_i, P_i, Q_i)$ is pairwise $\alpha$-compact.*

*Proof.* It is clear from theorem 2.2.

The converse of the above theorem, namely that the product of any family of pairwise $\alpha$-compact fuzzy bitopological spaces, is again pairwise $\alpha$-compact would be a generalization of Tychonoff's product theorem. But this is not the case as the following example shows.

EXAMPLE Let $X_i$ $(i=1, 2)$ be a countable set : $X_i = \{x_1, x_2, x_3, \ldots\}$. Let $P$ and $Q$ be the discrete fuzzy space and indiscrete fuzzy space. Then *fbs* $(X_1, P, Q)$ and *fbs* $(X_2, Q, P)$ are pairwise $\alpha$-compact since every pairwise $\alpha$-shading must contain the fuzzy set $\mathbf{1}$. Let $u_i$, $v_i \in P$ be the fuzzy open sets defined as follows. $u_i(x_i) > \alpha$, $u_i(x_j) = 0$ $i \neq i$, $v_i(x_i) > \alpha$, $v_i(x_j) = 0$ $i \neq j$. Then the pairwise $\alpha$-shading $\{p_1^{-1}(u_1), p_1^{-1}$ $(u_2), \ldots, p_2^{-1}(v_1), p_2^{-1}(v_2), \ldots\}$ of $(X_1 \times X_2, P \times Q, Q \times P)$ does not have a finite $\alpha$-subshading. Therefore $(X_1 \times X_2, P \times Q, Q \times P)$ is not pairwise $\alpha$-compact.

5.5. THEOREM *Let $\{X_i, P_i, Q_i)\}$ be any family of fuzzy bitopological spaces. Then $(\pi X_i, P, Q)$ is pairwise $\alpha$-connected iff each $(X_i, P_i, Q_i)$ is pairwise $\alpha$-connected.*

*Proof.* Suppose $(\Pi X_i, P, Q)$ is pairwise $\alpha$-connected. Then the projection $p_k : (\Pi X_i, P, Q) \rightarrow (X_k, P_k, Q_k)$ is a pairwise $F$-continuous surjection. Thus by theorem 4. 3. $(X_k, P_k, Q_k)$ is pairwise $\alpha$-connected for each $k$. Conversely, if $x=\{x_k\}$ and $y=\{y_k\}$ differ by at most finitely many coordinates, then $x$ and $y$ lie in a pairwise $\alpha$-connected fuzzy subspace of $(\Pi X_i, P, Q)$; this follows by induction on theorem 4. 1. and the fact that fuzzy injections preserve pairwise $\alpha$-connectivity. Futhermore, given $x=\{x_k\}$, then $P\text{-Cl}_\alpha(D)=\Pi X_i=Q\text{-Cl}_\alpha(D)$ where $D=\{y=\{y_k\}$ ; $x$ and $y$ differ by at most finitely many coordinates$\}$. To show this, let $z\in\Pi X_i$ and let $u\in P$ such that $u(z)>\alpha$, then there are $\gamma_1, \ldots, \gamma_n$ such that $u\geq \bigwedge_{i=1}^{n} p_{\gamma i}{}^{-1}(u_{\gamma i})$ and $\bigwedge_{i=1}^{n} p_{\gamma i}{}^{-1}(u_{\gamma i})(z)>\alpha$ where $u_{\gamma i}\in p_{\gamma i}$ for each $i$. Let $y=\{y_i\}$ be chosen such that $y_{\gamma i}=z_{\gamma i}$ for each $i$ and $y_\gamma=x_\gamma$ otherwise. Then $y\in D$ and $u(y)\geq \bigwedge_{i=1}^{n} p_{\gamma i}(u_{\gamma i})(y)=\bigwedge_{i=1}^{n} u_{\gamma i}(z_{\gamma i})=\bigwedge_{i=1}^{n} p_{\gamma i}{}^{-1}(u_{\gamma i})(z)>\alpha\geq 0$, hence $z\in P\text{-Cl}_\alpha(D)$.

Similarily, we have $z\in Q\text{-Cl}_\alpha(D)$, therefore $\Pi X_i=P\text{-Cl}_\alpha(D)\cap Q\text{-Cl}_\alpha(D)$. The theorem follows from theorem 4. 1.

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