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SOME GENERALIZATIONS OF COMPACT CONVERGENCE SPACES

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§0. Introduction

In the study of topology, the concept of compactness is generalized in many ways, namely the concept of real compactness or more generally that of k-compactness [5], that of minimal Hausdorff spaces, that of H-closed spaces, and that of paracompact spaces among others. Compact spaces are characterized by the fact that every ultrafilter is convergent. Hence compact convergence spaces are defined by the exactly same fact.

In this paper, we try to generalize the concept of compact convergence spaces in the similar vein. We introduce the concept of ultra real compact spaces by the fact that every ultrafilter with the countable intersection property is convergent and then get some properties of those spaces. We show that they are closed under the formation of products and closed subspaces. It is known [4] that every completely regular space has the real compactification, and [8] that every Hausdorff convergence space has a compactification. Using the same method, we show that every convergence space has an ultra real compactification.

For the terminology of convergence space, we refer to [1], [3].

§1. Ultra real compact spaces

1.1 DEFINITION. Let k be an infinite cardinal. A family $\mathcal{F}$ of subsets of X is said to have the *k-intersection property* if every subfamily of $\mathcal{F}$ whose cardinal is less than k has the non-empty intersection.

We note that a family has the finite (resp. countable) intersection property iff it has the $\aleph_0$-($\aleph_1$-resp.) intersection property.

In the following, k will denote an infinite cardinal.

1.2 DEFINITION. A convergence space is said to be *ultra k-compact* if every ultrafilter with the k-intersection property is convergent.

1.3 REMARK. 1) A convergence space is compact iff it is ultra $\aleph_0$-compact.

2) If there is no measurable cardinal, then a discrete topological space is ultra $\aleph_1$-compact (see [4]).

3) If k, n are infinite cardinal with $k \leq n$, then every ultra k-compact convergence space is ultra n-compact. In particular, for any infinite cardinal k, compact space is ultra k-compact.

4) Since an ultrafilter with the countable intersection property has the k-intersection property for any non-measurable cardinal k (see [4]), every ultra-k-compact space is also ultra $\aleph_1$-compact.

Thus if there is no measurable cardinal, then one has only two classes of ultra k-compact spaces, namely the class of compact spaces and that of ultra $\aleph_1$-compact spaces.

1.4 DEFINITION. An ultra $\aleph_1$-compact convergence space will be called an *ultra realcompact space.*

1.5 THEOREM. *An arbitrary product of ultra realcompact space is ultra realcompact.*

Proof. Let $X = \prod_{i \in I} X_i$, where each X_i is ultra realcompact, and let $Pr_i : X \to X_i$ be the i-th projection. Take any ultrafilter $\mathcal{F}$ on X with the countable intersection property, then for each $i \in I$, $Pr_i(\mathcal{F})$ is again an ultrafilter base with the countable intersection property. Since X is ultra realcompact, $Pr_i(F)$ is convergent, say to x_i. Hence $\mathcal{F}$ converges to (x_i) in X.

1.6 THEOREM. *A closed subspace of an ultra realcompact space is again ultra realcompact.*

Proof. Let F be a closed subspace of an ultra realcompact space X and $\mathcal{F}$ an ultrafilter on F with the countable intersection property. Since $\mathcal{F}$ is an ultrafilter base on X with the countable intersection property, it is convergent on X, say to x. Since $x \in \bar{F} = F$ and F is a subspace of X, $\mathcal{F}$ also converges to x on F.

1.7 DEFINITION Let $f : X \to Y$ be a continuous map between convergence spaces X and Y. Then f is said to be *perfect* if for any ultrafilter $\mathcal{U}$ on X such that $f(\mathcal{U})$ converges to y on Y, there is $x \in X$ such that $\mathcal{U}$ converges to x and $f(x) = y$.

1.8 THEOREM. *If $f : X \to Y$ is a perfect map and Y is ultra realcompact, then X is also ultra realcompact.*

Proof: For any ultrafilter $\mathcal{U}$ on X with the countable intersection property, $f(\mathcal{U})$ is also an ultrafilter base with the countable intersection property. Hence $f(\mathcal{U})$ is convergent. Since f is perfect, $\mathcal{U}$ is also convergent. This completes the proof.

For any Hausdorff convergence X, let $\{\mathfrak{F}_t | t\in T\}$ be the set of all ultrafilters on X with the countable intersection property, where $X\subseteq T$ and for $x\in X$, $F_x=\dot{x}$.

Let T denote the strict extension of X associated with $\{F_t | t\in T\}$ (see [7]).

1.9 THEOREM. *The space T is Hausdorff ultra real compactification of X, in other words, T is a Hausdorff extension of X which is ultra realcompact.*

Proof. Since T is a Hausdorff extension of X ([7]), it remains to show that T is ultra realcompact. Let Φ be an ultrafilter with the countable intersection property and let $\mathcal{U}=\{F\subseteq X| \hat{F}\in\Phi\}$, while $\hat{F}=\{t\in T| F\in\mathfrak{F}_t\}$ (see again [7] for the detail). Using Proposition 1.8 in [7], it is immediate that $\mathcal{U}$ is an ultrafilter on X. Take any sequence $\{F_n\}_{n\in N}$ in $\mathcal{U}$. Since Φ has the countable intersection property, $\cap\hat{F}_n\neq\phi$, say $t\in\cap\hat{F}_n$. Since $\mathfrak{F}_t$ has the countable intersection property and $F_n\in\hat{\mathfrak{F}}_t$ $(n\in\mathbf{N})$, $\cap F_n\neq\phi$. Thus $\mathcal{U}$ is an ultrafiler on X with the countable intersection property and hence there is some $t_0\in T$ with $\mathcal{U}=\mathfrak{F}_{t_0}$. For any $F\in\mathcal{U}=\mathfrak{F}_{t_0}$, $\hat{F}\in\Phi$, i.e., $\hat{\mathfrak{F}}_{t_0}\subseteq\Phi$ so that Φ converges to t_0. This completes the proof.

1.10 NOTATION. The space T in Theorem 1.8 will be denoted by $\mathcal{Q}_c X$.

1.11 THEOREM. *For any regular Hausdorff ultra realcompact space K and any continuous map $f: X\to K$, there is a continuous map $\bar{f}: \mathcal{Q}_c X\to K$ with $\bar{f}|X=f$.*

Proof: For any $t\in\mathcal{Q}_cX-X$, $f(\mathfrak{F}_t)$ is an ultrafilter base on K with the countable intersection property. Since K is Hausdorff ultra realcompact, $f(\mathfrak{F}_t)$ has a unique limit, say y_t. We define $\bar{f}: \mathcal{Q}_cX\to K$ by $\bar{f}(x)=f(x)$ $(x\in X)$ and $\bar{f}(t)=y_t$ $(t\in\mathcal{Q}_cX-X)$. Obviously $\bar{f}|X=f$ and hence it remains to show the continuity of $\bar{f}$. To do so, we observe that for any $A\subseteq X$, $\bar{f}(\hat{A})$ is contained in $\overline{f(A)}$. Indeed, for $x\in\hat{A}\cap X=A$, $\bar{f}(x)=f(x)\in f(A)\subseteq\overline{f(A)}$; for $t\in\hat{A}-X$, $A\in\mathfrak{F}_t$ and $f(\mathfrak{F}_t)\to\bar{f}(t)$ and hence $\bar{f}(t)\in\overline{f(A)}$. Now take any filter Φ on $\mathcal{Q}_cX$ converging to t. If $t\in X$, then there is a filter $\mathcal{G}$ on X converging to t and $\hat{\mathcal{G}}\subseteq\Phi$. Since f is continuous, $f(\hat{\mathcal{G}})\to f(t)$. Since Y is regular, $\overline{f(\mathcal{G})}\to f(t)=\bar{f}(t)$ and $\overline{f(\mathcal{G})}=\bar{f}(\mathcal{G})\subseteq\bar{f}(\hat{\mathcal{G}})\subseteq\bar{f}(\Phi)$ by the above observation. Thus $\bar{f}(\Phi)$ also converges to $\bar{f}(t)$. If $t\in\mathcal{Q}_cX-X$, then $\hat{\mathfrak{F}}_t\subseteq\Phi$. Since $f(\mathfrak{F}_t)\to\bar{f}(t)$ and $\overline{f(\mathfrak{F}_t)}=\overline{\bar{f}(\mathfrak{F}_t)}\subseteq\bar{f}(\hat{\mathfrak{F}}_t)\subseteq\bar{f}(\Phi)$, $\bar{f}(\Phi)$ also converges to $\bar{f}(t)$. This completes the proof.

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