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A NOTE ON A GENERIC SUBMANIFOLDS OF QUATERNIONIC PROJECTIVE SPACE

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§0. Introduction

Recently many authors have been studied some necessary and sufficient conditions or sufficient conditions to be one of model hypersurface $M_{p,q}{}^{Q}(a, b)$ in quaternionic projective space QP^m and developed those methods into generic submanifolds immersed in QP^m by using the theory of Riemannian fibre bundle (cf. Kon [9], Lawson [3], Pak[5], Shibuya [7], Yano[9]). In this point of view, present authors studied another sufficient conditions which are derived from locally symmetry of $\tilde{\pi}^{-1}(M)$ to determine certain generic submanifolds, where $\tilde{\pi}$ is the submersion defined by the Hopf-fibration: $S^{4m+3}\to QP^m$.

§1. The structure of a generic submanifold of QP^m

It is well known that a quaternionic projective space QP^m admits quaternionic Kaehlerian structure is a Kaehlerian manifold of constant Q-sectional curvature 4. (See Ishihara [1], [2] and Konish [2]).

Let QP^m be covered by a system of coordinate neighborhoods $\{\tilde{U}, y^h\}$ (in the sequel, the indices h, i, j run $\{1, 2, \dots, 4m\}$) and $F_i{}^h, G_i{}^h$ and $H_i{}^h$ the components of cannonical local base $\{F, G, H\}$ of 3-dimensional vector bundle V and g_{ji} those metric tensor. And let's denote by $K_{kji}{}^h$ components of the curvature tensors of QP^m. Since the unit sphere S^{4m+3} is a space of constant curvature 1. If we use the equation of co-Gauss, we find

$$(1.1)\quad K_{kji}{}^h=\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki}+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_{kj}F_i{}^h +G_k{}^h G_{ji}-G_j{}^h G_{ki}-2G_{kj}G_i{}^h+H_k{}^h H_{ji}-H_j{}^h H_{ki}-2H_{kj}H_i{}^h.$$

A submanifold M of QP^m is called a generic submanifold, if the normal space $N_p(M)$ of M at p is always mapped into tangent space $T_p(M)$ at p under the action of the cannonical local base F, G and H.

We consider an n-dimensional generic submanifold M of QP^m covered by a system of coordinate neighborhoods $\{U : x^a\}$ and represented by $y^i=y^i(x^a)$. And we denote the vectors $\partial_a y^i$ $(\partial_a=\partial/\partial x^a)$ tangent to M by $B_a{}^i$ and unit normal vectors by $N_x{}^i$ (In the sequel, the indices $x, y, z, \dots$ run $\{n+1, \dots, n+p\}$, $p=4m-n$). Hence, if we put in $\{U : x^a\}$

$$
\begin{aligned}
&F_h{}^iB_a{}^h=\phi_a{}^bB_b{}^i+\phi_a{}^xN_x{}^i,\quad F_h{}^iN_x{}^h=-\phi_x{}^aB_a{}^i,\\
(1.2)\qquad &G_h{}^iB_a{}^h=\psi_a{}^bB_b{}^i+\psi_a{}^xN_x{}^i,\quad G_h{}^iN_x{}^h=-\psi_x{}^aB_a{}^i,\\
&H_h{}^iB_a{}^h=\theta_a{}^bB_b{}^i+\theta_a{}^xN_x{}^i,\quad H_h{}^iN_x{}^h=-\theta_x{}^aB_a{}^i,
\end{aligned}
$$

then we get following structure, so called framed f-three structure, by applying F, G and H to (1, 2) and taking account of a quaternionic Kaehlerian structure (cf. [5], [6])

$$
\begin{aligned}
&\phi_c{}^b\phi_a{}^c=-\delta_a{}^b+\phi_a{}^x\phi_x{}^b,\quad \phi_a{}^b\phi_b{}^x=0\\
&\psi_c{}^b\psi_a{}^c=-\delta_a{}^b+\psi_a{}^x\psi_x{}^b,\quad \psi_a{}^b\psi_b{}^x=0\\
&\theta_c{}^b\theta_a{}^c=-\delta_a{}^b+\theta_a{}^x\theta_x{}^b,\quad \theta_a{}^b\theta_b{}^x=0\\
(1.3)\qquad &\phi_c{}^b\psi_a{}^c=-\theta_a{}^b+\phi_a{}^x\psi_x{}^b,\quad \theta_c{}^b\phi_a{}^c=\psi_a{}^b+\phi_a{}^x\theta_x{}^b\\
&\theta_c{}^b\psi_a{}^c=-\phi_a{}^b+\psi_a{}^x\theta_x{}^b,\quad \phi_c{}^b\psi_a{}^c=\theta_a{}^b+\psi_a{}^x\phi_x{}^b\\
&\phi_c{}^b\theta_a{}^c=-\psi_a{}^b+\theta_a{}^x\phi_x{}^b,\quad \psi_c{}^b\theta_a{}^c=\phi_a{}^b+\theta_a{}^x\psi_x{}^b\\
&\phi_a{}^c\psi_c{}^x=-\theta_a{}^x,\quad \phi_a{}^c\theta_c{}^x=\psi_a{}^x,\quad \psi_a{}^c\theta_c{}^x=-\phi_a{}^x\\
&\psi_a{}^c\phi_c{}^x=\theta_a{}^x,\quad \theta_a{}^c\phi_c{}^x=-\psi_a{}^x,\quad \theta_a{}^c\psi_c{}^x=\phi_a{}^x\\
&\phi_x{}^a\psi_a{}^y=0,\quad \psi_x{}^a\theta_a{}^y=0,\quad \theta_x{}^a\phi_a{}^x=0\\
&\phi_x{}^a\phi_a{}^y=\delta_x{}^y,\quad \psi_x{}^a\psi_a{}^y=\delta_x{}^y,\quad \theta_x{}^a\theta_a{}^y=\delta_x{}^y.
\end{aligned}
$$

We denote ∇_b be the covariant derivativative with respect to the Riemannian metric g_{ba} induced on M. Then equations of Gauss and Weingarten are given by

(1.4) $\nabla_bB_a{}^i=h_{ba}{}^xN_x{}^i,\quad \nabla_bN_x{}^i=-h_b{}^a{}_xB_a{}^i$

respectively, $h_{ba}{}^x$ being the components of the second fundamental tensor with respect to the unit normal vectors $N_x{}^i$, where

$$h_{bx}{}^a=g^{ae}h_{be}{}^yg_{yx},\quad (g^{ba})=(g_{ba})^{-1}\quad\text{and}\quad g_{yx}=g_{ji}N_y{}^jN_x{}^i.$$

Applying the operator $\nabla_c=B_c{}^j\nabla_j$ to (1.2) and taking account of quaternic Kaehlerian structure of QP^m and (1.4), we easily find that

$$
\begin{aligned}
&\nabla_c\phi_a{}^b=r_c\psi_a{}^b-q_c\theta_a{}^b+h_c{}^b{}_x\phi_a{}^x-h_{ca}{}^x\phi_x{}^b\\
(1.5)\qquad &\nabla_c\psi_a{}^b=-r_c\phi_a{}^b+p_c\theta_a{}^b+h_c{}^b{}_x\psi_a{}^x-h_{ca}{}^x\psi_x{}^b\\
&\nabla_c\theta_a{}^b=q_c\phi_a{}^b-p_c\psi_a{}^b+h_c{}^b{}_x\theta_a{}^x-h_{ca}{}^x\theta_x{}^b,
\end{aligned}
$$

where we have put $p_c=p_iB_c{}^i$, $q_c=q_iB_c{}^i$, $r_c=r_iB_c{}^i$.

Taking account of (1.4), we can see that the equations of Gauss, Codazzi and Ricci are respectively given by

$$
\begin{aligned}
(1.6)\quad K_{dcb}{}^a=&\delta_d{}^ag_{cb}-\delta_c{}^ag_{db}+\phi_d{}^a\phi_{cb}-\phi_c{}^a\phi_{db}-2\phi_{dc}\phi_b{}^a+\psi_d{}^a\psi_{cb}-\psi_c{}^a\psi_{db}\\
&-2\psi_{dc}\psi_b{}^a+\theta_d{}^a\theta_{cb}-\theta_c{}^a\theta_{db}-2\theta_{dc}\theta_b{}^a+h_d{}^a{}_xh_{cb}{}^x-h_c{}^a{}_xh_{db}{}^x.
\end{aligned}
$$

$$
\begin{aligned}
(1.7)\quad \nabla_ch_{ba}{}^x-\nabla_bh_{ca}{}^x=&\phi_c{}^x\phi_{ba}-\phi_b{}^x\phi_{ca}-2\phi_{cb}\phi_a{}^x+\psi_c{}^x\psi_{ba}\\
&-\psi_b{}^x\psi_{ca}-2\psi_{cb}\psi_a{}^x+\theta_c{}^x\theta_{ba}-\theta_b{}^x\theta_{ca}-2\theta_{cb}\theta_a{}^x.
\end{aligned}
$$

$$
\begin{aligned}
(1.8)\quad K_{cby}{}^x=&\phi_c{}^x\phi_{by}-\phi_b{}^x\phi_{cy}+\psi_c{}^x\psi_{by}-\psi_b{}^x\psi_{cy}+\theta_c{}^x\theta_{by}\\
&-\theta_b{}^x\theta_{cy}+h_{ca}{}^xh_b{}^a{}_y-h_{ba}{}^xh_c{}^a{}_y.
\end{aligned}
$$

where $K_{dcb}{}^a$ and $K_{cby}{}^x$ being components of the curvature tensors determined by the induced metric g_{cb} and g_{yx} in M and in the normal bundle of M, respectively.

§2. Generic submanifolds of a quaternionic Kaehlerian manifold with locally symmetric fibred Riemannian space

Covering $S^{4m+3}(1)$ by a system of coordinate neighborhoods $\{\hat{U} : y^{\kappa}\}$ such that $\tilde{\pi}(\hat{U})=\tilde{U}$ are coordinate neighborhoods of QP^m with local coordinate (y^j), we can represent the projection $\tilde{\pi} : S^{4m+3}\to QP(m)$ by $y^j=y^j(y^{\kappa})$ and put $E_{\kappa}{}^j=\partial_{\kappa}y^j$ $(\partial_{\kappa}=\partial/\partial y^{\kappa})$ with the rank of matrix $(E_{\kappa}{}^j)$ being always $4m$ (In the sequel, the indices κ, μ, ν run $\{1, 2, \ldots, 4m+3\}$).

Let's denote by ξ^{κ}, $\tilde{\eta}^{\kappa}$ and $\tilde{\zeta}^{\kappa}$ components of $\tilde{\xi}$, $\tilde{\eta}$ and $\tilde{\xi}$ of the induced Sasakian 3-structure $\{\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}\}$ in S^{4m+3} respectively.

Next we define $E^{\kappa}{}_j$ by $(E^{\kappa}{}_j,\ \tilde{C}_s{}^{\kappa})=(E_{\kappa}{}^j, \tilde{C}_{\kappa}{}^s)^{-1}$, then $\{E^{\kappa}{}_j,\ \tilde{C}^{\kappa}{}_s\}$ is a local frame in $\hat{U}$ and $\{E_{\kappa}{}^j, \tilde{C}_{\kappa}{}^s\}$ the frame dual to $\{E^{\kappa}{}_j, \tilde{C}^{\kappa}{}_s\}$, where

$$\tilde{C}^{\kappa}{}_s=a_s\tilde{\xi}^{\kappa}+b_s\tilde{\eta}^{\kappa}+c_s\tilde{\zeta}^{\kappa},\quad a_sa^t+b_sb^t+c_sc^t=\delta_s{}^t.$$

Now, we take coordinate neighborhoods $\{\bar{U} : x^{\alpha}\}$ of $\tilde{\pi}^{-1}(M)$ such that $\pi(\bar{U})=U$ are coordinate neighborhoods of M with local coordinates (x^a), where π is a compatible submersion with totally geodesic fibres.

Thus, if we let the isometric immersion $\tilde{i} : \tilde{\pi}^{-1}(M)\to S^{4m+3}$ be locally expressed by $y^{\kappa}=y^{\kappa}(x^{\alpha})$, then the commutativity $\tilde{\pi}\circ\tilde{i}=i\circ\pi$ implies that $\{E_{\alpha}{}^a, C_{\alpha}{}^s\}$ is a local coframe in $\tilde{\pi}^{-1}(M)$ corresponding to $\{E_{\kappa}{}^j, \tilde{C}_{\kappa}{}^s\}$ in S^{4m+3} and $\{E^{\alpha}{}_a, C^{\alpha}{}_s\}$ the coframe dual to $\{E_{\alpha}{}^a, C_{\alpha}{}^s\}$ (Where in the sequel, the indices α, β, γ and a, b, c run over $\{1, \ldots, n+3\}$ and $\{1, \ldots, n\}$ respectively). Since $\xi^{\alpha}, \eta^{\alpha}$ and ζ^{α} are vertical vectors and span the tangent space to the fibre $\mathcal{F}$ at each point of $\bar{M}=\tilde{\pi}^{-1}(M)$, we can put in $\bar{U}$

$$C^{\alpha}{}_s=a_s\xi^{\alpha}+b_s\eta^{\alpha}+c_s\zeta^{\alpha}, \tag{2.1}$$

$$a_sa^t+b_sb^t+c_sc^t=\delta_s{}^t, \tag{2.2}$$

where the functions a_s, b_s and c_s are the restrictions of a_s, b_s and c_s appearing in $\tilde{C}^{\kappa}{}_s$.

Let's denote the metrics on $\tilde{\pi}^{-1}(M)$ by $g_{\alpha\beta}=G_{\lambda\mu}B_{\alpha}{}^{\lambda}B_{\beta}{}^{\mu}$ where $G_{\lambda\mu}$ metrics on S^{4m+3}. Then van der Waerden-Bortolotti covariant derivative of $E_{\alpha}{}^a$, $E_a{}^{\alpha}$ are given by (See Ishihara and Konish [2])

$$\begin{aligned}
&\bar{\nabla}_{\varepsilon}E_{\alpha}{}^a=h_b{}^a{}_s(E_{\varepsilon}{}^bC_{\alpha}{}^s+C_{\varepsilon}{}^sE_{\alpha}{}^b),\\
&\bar{\nabla}_{\varepsilon}E^{\delta}{}_d=h_{bd}{}^sE_{\varepsilon}{}^bC^{\delta}{}_s-h_d{}^b{}_sC_{\varepsilon}{}^sE^{\delta}{}_b,\\
&\bar{\nabla}_{\varepsilon}C^{\delta}{}_s=-h_c{}^a{}_sE_{\varepsilon}{}^cE^{\delta}{}_a+P_{cs}{}^tE_{\varepsilon}{}^cC^{\delta}{}_t,\\
&\bar{\nabla}_{\varepsilon}C_{\delta}{}^s=-h_{cb}{}^sE_{\varepsilon}{}^cE_{\delta}{}^b-P_{ct}{}^sE_{\varepsilon}{}^cC_{\delta}{}^t,
\end{aligned} \tag{2.3}$$

where $h_b{}^a{}_s=-(a_s\phi_b{}^a+b_s\psi_b{}^a+c_s\theta_b{}^a)$, $\phi_b{}^a, \psi_b{}^a$ and $\theta_b{}^a$ are framed f-3-structure tensors which are given in (1.3).

When $\tilde{\pi}^{-1}(M)$ is locally symmetric space, if we apply $\bar{\nabla}_{\varepsilon}$ to $\bar{K}_{dcb}{}^a=\bar{K}_{\delta\gamma\beta}{}^{\alpha}E^{\delta}{}_dE^{\gamma}{}_cE^{\beta}{}_bE_{\alpha}{}^a$ and use (2.3), we get

$$\begin{aligned}
\bar{\nabla}_{\varepsilon}\bar{K}_{dcb}{}^a=&E_{\varepsilon}{}^e(h_{ed}{}^s\bar{K}_{scb}{}^a+h_{ec}{}^s\bar{K}_{dsb}{}^a+h_{eb}{}^s\bar{K}_{dcs}{}^a+h_e{}^a{}_s\bar{K}_{dcb}{}^s)\\
&-C_{\varepsilon}{}^s(h_d{}^e{}_s\bar{K}_{ecb}{}^a+h_c{}^e{}_s\bar{K}_{deb}{}^a+h_b{}^e{}_s\bar{K}_{dce}{}^a-h_e{}^a{}_s\bar{K}_{dcb}{}^e),
\end{aligned}$$

from which, transvecting $a^tC^\varepsilon{}_t=\xi^\varepsilon$, $b^tC^\varepsilon{}_t=\eta^\varepsilon$ and $c^tC^\varepsilon{}_t=\zeta^\varepsilon$ respectively, we find

(2.4) $\phi_d{}^e\tilde{K}_{ecb}{}^a+\phi_c{}^e\tilde{K}_{deb}{}^a+\phi_b{}^e\tilde{K}_{dce}{}^a-\phi_e{}^a\tilde{K}_{dcb}{}^e=0$

(2.5) $\psi_d{}^e\tilde{K}_{ecb}{}^a+\psi_c{}^e\tilde{K}_{deb}{}^a+\psi_b{}^e\tilde{K}_{dce}{}^a-\psi_e{}^a\tilde{K}_{dcb}{}^e=0$

(2.6) $\theta_d{}^e\tilde{K}_{ecb}{}^a+\theta_c{}^e\tilde{K}_{deb}{}^a+\theta_b{}^e\tilde{K}_{dce}{}^a-\theta_e{}^a\tilde{K}_{dcb}{}^e=0$

with the helf of projectivity of

$$K^H=K_{dcb}{}^aE^d\otimes E^c\otimes E^b\otimes E_a$$

and (2.1), (2.2).

Thus we have the following proposition;

PROPOSITION 2.1 *Let M be a generic submanifold of QP^m and $\pi: \bar{M}\to M$ the submersion which is compatible with the Hopf-fibration $\tilde{\pi}: S^{4m+3}\to QP^m$, then locally symmetric submanifold $\tilde{\pi}^{-1}(M)$ satisfies the following identities*

I) $g(K(\phi X)^L, Y^L)Z^L, W^L)+g(K(X^L, (\phi Y)^LZ^L, W^L)$
$\quad +g(K(X^L, Y^L)(\phi Z)^L, W^L)+g(K(X^L, Y^L)Z^L, (\phi W)^L)=0$

II) $g(K(\psi X)^L, Y^L)Z^L, W^L)+g(K(X^L, (\psi Y)^LZ^L, W^L)$
$\quad +g(K(X^L, Y^L)(\psi Z)^L, W^L)+g(K(X^L, Y^L)Z^L, (\psi W)^L)=0$

III) $g(K((\theta X)^L, Y^L)Z^L, W^L)+g(K(X^L, (\theta Y)^LZ^L, W^L)$
$\quad +g(K(X^L, Y^L)(\theta Z)^L, W^L)+g(K(X^L, Y^L)Z^L, (\theta W)^L)=0$

for any vector fields X, Y, Z, W and framed f-3-structure tensor $\{\phi, \psi, \theta\}$ on M, where X^L means horizontal lift of vector field X tangent to M.

On the other hand, if we take covariant derivative $\bar{\nabla}_\varepsilon$ *to* $\tilde{K}_{dcb}{}^s=\tilde{K}_{\delta\gamma\beta}{}^\alpha E^\delta{}_dE^\gamma{}_cE^\beta{}_bC_\alpha{}^s$ in locally symmetric submanifold $\tilde{\pi}^{-1}(M)$, we get by using

(2.3) $$\bar{\nabla}_\varepsilon\tilde{K}_{dcb}{}^t=E_\varepsilon{}^e\{h_{ed}{}^s\tilde{K}_{scb}{}^t+h_{ec}{}^s\tilde{K}_{dsb}{}^t+h_{eb}{}^s\tilde{K}_{dcs}{}^t-h_{ea}{}^t\tilde{K}_{dcb}{}^a-p_{es}{}^t\tilde{K}_{dcb}{}^c\}$$
$$-C_\varepsilon{}^s\{h_d{}^e{}_s\tilde{K}_{ecb}{}^t+h_c{}^e{}_s\tilde{K}_{deb}{}^t+h_b{}^e{}_s\tilde{K}_{dce}{}^t\},$$

from which, transvecting $C^\varepsilon{}_u$, we find

$$C^\varepsilon{}_u\bar{\nabla}_\varepsilon\tilde{K}_{dcb}{}^t=h_d{}^e{}_u\tilde{K}_{ecb}{}^t+h_c{}^e{}_u\tilde{K}_{deb}{}^t+h_b{}^e{}_u\tilde{K}_{dce}{}^t,$$

from which, transvecting a_ta^u, b_tb^u and c_tc^u respectively, and using equation of co-Codazzi, we get

(2.7) $$\phi_d{}^e\nabla_b\phi_{ce}+\phi_c{}^e\nabla_b\phi_{ed}+\phi_b{}^e\nabla_e\phi_{cd}=0$$

(2.8) $$\psi_d{}^e\nabla_b\psi_{ce}+\psi_c{}^e\nabla_b\psi_{ed}+\psi_b{}^e\nabla_e\psi_{cd}=0$$

(2.9) $$\theta_d{}^e\nabla_b\theta_{ce}+\theta_c{}^e\nabla_b\theta_{ed}+\theta_b{}^e\nabla_e\theta_{cd}=0$$

respectively, by virtue of

$$(\mathcal{L}_\xi\phi^H)^H=0,\quad (\mathcal{L}_\eta\phi^H)^H=-2\theta^H,\quad (\mathcal{L}_\xi\phi^H)^H=2\psi^H$$
$$(\mathcal{L}_\xi\psi^H)^H=2\theta^H,\quad (\mathcal{L}_\eta\psi^H)^H=0,\quad (\mathcal{L}_\xi\psi^H)^H=-2\phi^H$$
$$(\mathcal{L}_\xi\theta^H)^H=-2\psi^H,\quad (\mathcal{L}_\eta\theta^H)^H=2\phi^H,\quad (\mathcal{L}_\xi\theta^H)^H=0,$$

on fibre $\mathfrak{F}$ of $\tilde{\pi}^{-1}(M)$, (2.1) and (2.2), where $\{\xi, \eta, \xi\}$ are triple killing vectors.

Now substituting (1.5) into (2.7), (2.8) and (2.9), and transvecting $\phi_y{}^c\phi_z{}^b$, $\psi_y{}^c\psi_z{}^b$ and $\theta_y{}^c\theta_z{}^b$ respectively, we get

(2.10) $\phi_d{}^a A_{aby}\phi_z{}^b=(r_b\phi_z{}^b)\psi_{dy}-(q_b\phi_z{}^b)\theta_{dy}$,

(2.11) $\psi_d{}^a A_{aby}\psi_z{}^b=-(r_b\psi_z{}^b)\phi_{dy}+(p_b\psi_z{}^b)\theta_{dy}$,

(2.12) $\theta_d{}^a A_{aby}\theta_z{}^b=(q_b\theta_z{}^b)\phi_{dy}-(p_b\theta_z{}^b)\psi_{dy}$ by virtue of (1.3).

From which, transvecting ψ^{cd} to (2.10) and using (1.3), we have

$$\{-4(2m-p)+2p\}\ (q_b\phi_z{}^b)-2(\phi_c{}^a A_{ba}{}^x\phi_z{}^b)\theta_x{}^c=0.$$

Hence we get $8(m-p)q_b\phi_z{}^b=0$, by taking account of (2.10). Similarly, if we transvect θ^{cd} to (2.10) and also using (1.3), (2.10) itself, we find $8(m-p)r_b\phi_z{}^b=0$. Thus, applying above methods to (2.11), (2.12), respectively, we get

PROPOSITION 2.2. *Under the same assumptions in* Proposition 2.1. $(m\neq p)$, *M satisfies*;

IV) $\phi A^N\phi_M=0,\ \psi A^N\psi_M=0,\ \theta A^N\theta_M=0,$

where A^N and $\{\phi_M, \psi_M, \theta_M\}$ are second fundamental tensors and structure vectors with respect to normal vectors N^N $(N, M=1, \dots, p)$ respectively.

§3. Generic submanifolds of QP^m satisfying certain conditions

In previous section, we have introduced some properties of M derived from locally symmetry of $\tilde{\pi}^{-1}(M)$. In this section, we want to study converse problem. Then the generic submanifold of QP^m satisfying certain conditions will be determined.

Since $\overline{M}$ is a submanifold of S^{4m+3}, if we use equation of co-Gauss and (1.6) to curvature tensor of $\overline{M}$ then $\tilde{K}_{dcba}=\tilde{g}(K(E_d, E_c)E_b, E_a)$ are given as following form

(3.1) $\tilde{K}_{dcb}{}^a=\delta_d{}^a g_{cb}-\delta_c{}^a g_{db}+A_d{}^a{}_x A_{cb}{}^x-A_c{}^a{}_x A_{db}{}^x.$

Substituting (3.1) into (I), (II) and (III) imply following equations

(3.2) $(\phi_a{}^e A_{dex}+\phi_d{}^e A_{aex})A_{cb}{}^x+(\phi_b{}^e A_{cex}+\phi_c{}^e A_{ebx})A_{da}{}^x$
$-(\phi_b{}^e A_{dex}+\phi_d{}^e A_{ebx})A_{ca}{}^x-(\phi_a{}^e A_{cex}+\phi_c{}^e A_{aex})A_{db}{}^x=0,$

(3.3) $(\psi_a{}^e A_{dex}+\psi_d{}^e A_{aex})A_{cb}{}^x+(\psi_b{}^e A_{cex}+\psi_c{}^e A_{ebx})A_{da}{}^x$
$-(\psi_b{}^e A_{dex}+\psi_d{}^e A_{ebx})A_{ca}{}^x-(\psi_a{}^e A_{cex})A_{db}{}^x=0,$

(3.4) $(\theta_a{}^e A_{dex}+\theta_d{}^e A_{aex})A_{cb}{}^x+(\theta_b{}^e A_{cex}+\theta_c{}^e A_{ebx})A_{da}{}^x$
$-(\theta_b{}^e A_{dex}+\theta_d{}^e A_{ebx})A_{ca}{}^x-(\theta_a{}^e A_{cex}+\theta_c{}^e A_{aex})A_{db}{}^x=0.$

On the other side, transvecting $\phi_c{}^d, \psi_c{}^d$ and $\theta_c{}^d$ to (IV) respectively, we have

(3.5) $A_{db}{}^x\phi_z{}^b=P_{yz}{}^x\phi_d{}^y,\ A_{db}{}^x\psi_z{}^b=Q_{yz}{}^x\psi_d{}^y,\ A_{db}{}^x\theta_z{}^b=R_{yz}{}^x\theta_d{}^y,$

where we have put $P_{yz}{}^x=A_{be}{}^x\phi_z{}^b\phi_y{}^e$, $Q_{yz}{}^x=A_{be}{}^x\psi_z{}^b\psi_y{}^e$ and $R_{yz}{}^x=A_{be}{}^x\theta_z{}^b\theta_y{}^e$.

Transvecting $\phi_z{}^a$ to (3.2) and taking account of (IV) and (3.5), we easily find,

$$P_{yz}{}^{x}\phi_{d}{}^{y}(\phi_{b}{}^{e}A_{cex}+\phi_{c}{}^{e}A_{ebx})-P_{yz}{}^{x}\phi_{c}{}^{y}(\phi_{b}{}^{e}A_{dex}+\phi_{d}{}^{e}A_{bex})=0.$$

From which, transvecting $\phi_{w}{}^{c}$, we have by using (1.3) and (IV),

(3.6) $P_{wz}{}^{x}(\phi_{b}{}^{e}A_{dex}+\phi_{d}{}^{e}A_{bex})=0.$

Similarly, applying those methods to (3.3) and (3.4), respectively, we also find,

(3.7) $Q_{wz}{}^{x}(\psi_{b}{}^{e}A_{dex}+\psi_{d}{}^{e}A_{bex})=0,\ R_{wz}{}^{x}(\theta_{b}{}^{e}A_{dex}+\theta_{d}{}^{e}A_{bex})=0.$

Now suppose the n-dimensional generic submanifold M of $QP^{n+p/4}$ has flat normal connections, then we have by (1.8)

$$A_{be}{}^{x}A_{a}{}^{e}{}_{y}-A_{ae}{}^{x}A_{b}{}^{e}{}_{y}+\phi_{b}{}^{x}\phi_{ay}-\phi_{a}{}^{x}\phi_{by}-\psi_{b}{}^{x}\psi_{ay}-\psi_{a}{}^{x}\psi_{by}+\theta_{b}{}^{x}\theta_{ay}-\theta_{a}{}^{x}\theta_{by}=0.$$

Hence, if we transvect $\phi_{z}{}^{a}\phi_{v}{}^{b}$, $\psi_{z}{}^{a}\psi_{v}{}^{b}$, $\theta_{z}{}^{a}\theta_{v}{}^{b}$ to above equation, we find

$$(3.8)\qquad \begin{aligned} &P_{zy}{}^{u}P_{uv}{}^{x}-P_{zu}{}^{x}P_{vy}{}^{u}+\delta_{v}{}^{x}g_{yz}-\delta_{z}{}^{x}g_{yv}=0,\\ &Q_{zy}{}^{u}Q_{uv}{}^{x}-Q_{zu}{}^{x}Q_{vy}{}^{u}+\delta_{v}{}^{x}g_{yz}-\delta_{z}{}^{x}g_{yv}=0,\\ &R_{zy}{}^{u}R_{uv}{}^{x}-R_{zu}{}^{x}R_{vy}{}^{u}+\delta_{v}{}^{x}g_{yz}-\delta_{z}{}^{x}g_{yv}=0. \end{aligned}$$

Therefore we conclude that

$$(3.9)\qquad \begin{aligned} &(\phi_{a}{}^{e}A_{dev}+\phi_{d}{}^{e}A_{aev})g_{yz}-(\phi_{a}{}^{e}A_{dez}+\phi_{d}{}^{e}A_{aez})g_{yv}=0,\\ &(\psi_{a}{}^{e}A_{dev}+\psi_{d}{}^{e}A_{aev})g_{yz}-(\psi_{a}{}^{e}A_{dez}+\psi_{d}{}^{e}A_{aez})g_{yv}=0,\\ &(\theta_{a}{}^{e}A_{dev}+\theta_{d}{}^{e}A_{aev})g_{yz}-(\theta_{a}{}^{e}A_{dez}+\theta_{d}{}^{e}A_{aez})g_{yv}=0 \end{aligned}$$

by virture of (3.6), (3.7) and (3.8), where $g_{yv}=g_{cb}N_{y}{}^{c}N_{v}{}^{b}$ being the metric tensor of the normal bundle of M.

Contracting equation of (3.9) with respect to y and z, we get

$$\phi_{a}{}^{e}A_{dex}+\phi_{d}{}^{e}A_{aex}=0,\quad \psi_{a}{}^{e}A_{dex}+\psi_{d}{}^{e}A_{aex}=0,$$
$$\theta_{a}{}^{e}A_{dex}+\theta_{d}{}^{e}A_{aex}=0,$$

for $p>1$ When $p=1$, one of present authors showed those implications in[8] and determined M which has above corresponding conditions in a real hypersurface of QP^{m} was model space $M_{p,q}{}^{Q}(a,b)$.

Thus we have

THEOREM 3.3. *Let M be an n-dimensional generic submanifold of $QP^{n+p/4}$ with flat normal connection.* ($n\neq 3p$). *If M satisfies* (3.2), (3.3), (3.4) *and* IV), *then framed f-three-structure tensors* $\{\phi,\psi,\theta\}$ *of M commutes with its 2nd fundamental tensor on M.*

From this fact and Theorems in [5], we have

THEOREM 3.4. *Let M be a complete, generic submanifold of dimension n in quaternic projective space $QP^{n+p/4}$ ($n\neq 3p$) with flat normal connection. Suppose M satisfy* (3.2), (3.3), (3.4) *and*(IV), *and has parallel mean curvature vector in the normal bundle, then M is of the form*

$$\tilde{\pi}(S^{p_1}(r_1)\times\ldots\times S^{p_N}(r_N)),$$

where $p_1,\ldots,p_N\geq 1$, $P_i=4l_i+3$ (l_i; *non-negative integer*), $\sum_i r_i{}^2=1$, $\sum_i p_i=n+3$, $N=p+1$.

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