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BOL LOOPS FROM GROUP EXTENSIONS, a further note

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0. Introduction

In [3] Bol loops are constructed from split extensions of groups. In this note it is investigated whether non split extensions of groups can give rise in a similar manner to Bol loops. It is shown that a generalisation of the construction given in [3] gives rise to power associative loops which are not in general Bol loops. The condition obtained for the corresponding loop to be Bol is almost identical to that which is necessary and sufficient for the loop constructed to be embedded in the Bol wreath product mentioned in [3]. This generalised construction gives hopes for the construction of new Bol loops of small orders and Bol loops without Sylow subloops. The construction of Bol split extensions in this note differs slightly from that in [3] but it is obviously equivalent and ties in more closely with the ideas given in [2]. The construction given here is limited to the case where the kernel of the extension is abelian.

1. Left Bol extensions constructed from non-split group extensions

An arbitrary group extension E_1 of a group G by a G-module A may be represented by the following multiplication on the set $G\times A$; let $x, y\in G$, $a, b\in A$. Then

$$(x, a)(y, b)=(xy, f(x, y)+a\beta(y)+b) \tag{1}$$

Here $f: G\times G\to A$ is a cocycle and β is a homomorphism from G to $Aut(A)$. We consider the extension E_2 defined by

$$(x, a)(y, b)=(xy, f(y, x)+a+b\beta(x)) \tag{2}$$

The process of deriving E_2 from E_1 may be regarded as a "reflection" as follows. In [2] an arbitrary loop extension of a loop Q by a "loop module" A is described using the multiplication on $Q\times A$ as follows; let $x, y\in Q$, $a, b\in A$. Then

$$(x, a)(y, b)=(xy,\ f(x, y)+aL(x, y)+bR(x, y)) \tag{3}$$

where L and R are maps from $Q\times Q$ to $Aut(A)$ which satisfy the normalisation conditions $L(x, e)=R(x, e)=R(e, x)=id.$ for all $x\in Q$, and f is a map from $Q\times Q$ to A. Thus the extension E_2 is obtained from the maps f_1, L_1, R_1 which define E_1 by forming the maps f_2, L_2, R_2 defined by

$$f_2(x, y)=f_1(y, x),\ L_2(x, y)=R_1(y, x),\ R_2(x, y)=L_1(y, x),$$

for all x, y in G. Note that E_2 is not strictly a loop extension as defined in [2] since the normalisation conditions need not hold.

It is straightforward to verify that E_2 is associative if for all x, y, z in G $\beta(xy)=\beta(yx)$ and

$$f(yz, x)+f(z, y)\beta(x)=f(z, xy)+f(y, x) \tag{4}$$

This may be compared with the usual cocycle identity for f

$$f(x, y)\beta(z)+f(xy, z)=f(x, yz)+f(f, z) \tag{5}$$

In partiular if G is abelian then E_2 is associative. If however G is nonabelian (4) need not be satisfied. In fact even if f is a coboundary, i.e. E_1 is a split extension (4) need not be satisfied. For let $f(x, y)=c(x)\beta(y)+c(y)-c(xy)$, where c is an arbitrary map from G to A. Then (4) is equivalent to

$$\begin{aligned}&(c(yz)-c(zy))\beta(x)-c(yzx)+c(z)\beta(y)\beta(x)\\ &=c(z)\beta((xy)+c(xy)-c(yx)-c(zxy)\end{aligned} \tag{6}$$

In particular if x lies in the centre of G then (6) reduces to

$$(c(yz)-c(zy))\beta(x)+c(yzx)-c(zyx)=0 \tag{7}$$

Thus if c is chosen so that $c(yz)=c(zy)$ but $c(yzx)\neq c(zyx)$ then (4) is not satisfied. Thus cohomologous factor sets of E_1 can give rise to inequivalent extensions via the reflection process.

If E_1 arises as the pullback of an extension H of G/G' by A, i.e. if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & G & \longrightarrow & e \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\phi} & & \\ 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & G/G' & \longrightarrow & e \end{array}$$

where the rows are exact and ϕ is the natural surjection then there will be a factor set f of E_1 which satisfies $f(xk_1, yk_2)=f(x, y)$ for all x, y in G, k_1, k_2 in G'. In this case (4) is automatically satisfied.

The left Bol identity $x(y\cdot xz)=(x\cdot yx)z$ is satisfied by E_2 if and only if

$$\begin{aligned}&f(yxz, x)[f(xz, y)+b+[f(z, x)+z+c\beta(x)]\beta(y)]\beta(x)\\ &=f(x, xyx)+f(yx, x)+a+[f(x, y)+b+a\beta(y)]\beta(x)\end{aligned} \tag{8}$$

Since E_1 satisfies the right Bol identity we have

$$f(z, xyx)+c\beta(xyx)+f(xy, x)+[f(x, y)+a\beta(y)+b]\beta(x)+a$$
$$=f(zxy, x)+[f(zx, y)+[f(z, x)+c\beta(x)+a]\beta(y)+b]\beta(x)+a \qquad (9)$$

Thus if $f(xk, y)=f(x, y)$ for all x, y in G, k in G' then (8) and (9) are equivalent, i.e., E_2 is left Bol. We summarise:

THEOREM 1. *Given a factor set f of a group extension E_1 of a group G by a group module A a loop extension E_2 may be produced by the operation defined in* (2). *In general E_2 is non-associative.*

If f satisfies $f(xk, y)=f(x, y)$ for all x, y in G, k in G' then E_2 is left Bol. In particular such left Bol extensions may be constructed from extensions E_1 which are pullbacks of extensions of G/G' by A.

We note that it is easy to ensure that extensions E_2 are non associative (and non-Moufang) by taking extensions E_1 such that $\beta(xy)\neq\beta(yx)$. Multiplication in E_2 is power associative and it is straightforward to verify that the order of the element (x, a) is the same under the operations defined by (1) and (2). It is not obvious that Sylow subloops exist exist except in the case where f is trivial (see [3]). No explicit example is known of a factor set which satisfies $f(xk, y)=f(x, y)$ for all x, y in $G, k\in G'$ and which fails to satisfy $f(x, yk)=f(x, y)$ for some x, y in $G, k\in G'$. However there is strong circumstantial evidence of the existence of such factor sets in that they occur in the filtration given on p. 119 of [1].

2. Embedding in the wreath product

Since the wreath product of a group G by a group H is a split extension, the construction in [3] gives rise to a "Bol wreath product". More specifically the Bol wreath product of G by H, $G\sim_B H$ is the set $\{(g, \phi)\}$ where $\phi: G\to H$ is an arbitrary function and $g\in G$, with the following multiplication;

$$(g_1, \phi_1)(g_2, \phi_2)=(g_1, g_2, F),$$

where $F(x)=\phi_1(x)\phi_2(g_1x)$ for x, g_1, g_2 in G.

Now let H be a G-module A and let E_2 be defined by (2). We consider the embedding $\tau: E_2\to G\sim_B A$ defined by $\tau(g, a)=(g, \phi)$ where $\phi(x)=a\beta(x)+f(g, x)$, for $x\in G$.

THEOREM 2. *The embedding τ is a homomorphism if and only if f satisfies the identity $f(hg, x)=f(gh, x)$ for all h, g, x in G.*

Proof. The proof is by direct verification. Let $Y=(g, a)$, $Z=(h, b)$. Then

$\tau(Y)=(g,\phi)$ and $\tau(Z)=(h,\psi)$ where $\phi(x)=a\beta(x)+f(g,x)$ and $\psi(x)=b\beta(x)+f(h,x)$ where $x,g,h\in G$. Thus $\tau(Y)\cdot\tau(z)=(gh,F_1)$ where

$$F_1(x)=\phi(x)\psi(gx)=a\beta(x)+f(g,x)+b\beta(x)+f(h,gx).$$

Now $YZ=(gh,f(h,gx)+b\beta(g))$, and then

$$\tau(YZ)=(gh,\ (f(h,g)+a+b\beta(g))\beta(x)+f(gh,x)).$$

Thus τ is a homomorphism iff

$$f(h,g)\beta(x)+f(gh,x)=f(g,x)+f(h,gx) \qquad (10)$$

The conclusion of the theorem follows since τ is a homomorphism iff (5) and (10) are simultaneously satisfied.

COROLLARY. *If $f(xk,y)=f(x,y)$ for all $k\in G'$, x,y in G then E_2 may be embedded in $G\sim_B A$.*

3. Remarks

1. This extension of the construction given in [3] illustrates the close tie-up between group extensions which give rise to groups and those which give rise to Bol loops. Non-associative Moufang loops do not arise from this construction.

2. The reflection process as described in §1 suggests the following question: which varieties of loops are closed under this process? In other words, which varieties $\mathfrak{V}$ of loops are such that whenever an extension of a loop Q by a loop module A lies in $\mathfrak{V}$ then the reflected extension (with respect to any cocycle) lies in $\mathfrak{V}$? The only obvious examples are the variety of all loops and the variety of commutative loops.

References

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