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PROPERTIES OF *RS*-COMPACT SPACES

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0. Introduction

In [3] the concept of an *RS*-compact space was defined. In this paper, we show that the product space of *RS*-compact spaces is not *RS*-compact and prove that each *RS*-compact metrizable space is finite. Finally, we investigate the irresolute image of any weak *RS*-compact Hausdorff space in any Hausdorff space is closed.

In order for this paper to be as self-contained as possible, the following definitions are given. A subset U of a topological space is *regular semi-open* [1] if there exists a regular open set V such that $V\subset U\subset \mathrm{Cl}(V)$. A topological space X is *RS-compact* (*weak RS-compact*) if every regular semi-open cover has a finite subfamily whose interiors cover X (resp. if every regular semi-open cover has a finite subcover). A topological space is *extremally disconnected* [8] if the closure of every open set is open. A subset S of a topological space is *semi-open* [4] if there exists an open set O such that $O\subset S\subset Cl(O)$. A function $f: X\to Y$ is said to be *irresolute* [4] if the inverse image of every semi-open set is semi-open.

1. Products of RS-compact spaces

In Theorem 2.4 of [3], we proved that "Let $\{X_i | i\in I\}$ be a family of topological spaces. If $\Pi_i X_i$ is *RS*-compact, then X_i is *RS*-compact for each $i\in I$." In this section we prove that the inverse of Theorem 2.4 is false as shown by the following example.

THEOREM 1.1. *Every extremally disconnected, compact space is RS-compact.*

Proof. Suppose that $\{U_i\}_{i\in I}$ is a regular semi-open cover of an extremally disconnected and compact space X. Then there is a regular open set V_i such that $V_i\subset U_i\subset \mathrm{Cl}(V_i)$ for each $i\in I$. Since X is extremally disconnected and each $V_i=\mathrm{Int}\ (\mathrm{Cl}(V_i))$, $U_i=\mathrm{Int}(U_i)$ for each $i\in I$. Thus X is *RS*-compact since X is compact.

COROLLARY 1.2. *βN is RS-compact.*

Proof. βN is compact and extremally disconnected [z]. By Theorem 1

βN is RS-compact.

Combining the results of Theorem 1.1 [3] and Theorem 7 [6], we have the following theorem.

THEOREM 1.3. *If X is an RS-compact Hausdorff space, then X is extremally disconnected.*

EXAMPLE. βN is RS-compact and extremally disconnected. Although $\beta N\times\beta N$ is Hausdorff, it is not extremally disconnected [2, p. 97]. Therefore, by Theorem 1.3, $\beta N\times\beta N$ is not RS-compact. It is thus show that the product of two RS-compact spaces is not necessarily RS-compact.

2. *RS*-compact metrizable spaces

LEMMA 2.1. *Each RS-compact and semiregular space is compact.*

Proof. Let $\{O_i | i\in I\}$ be an open cover of the space X. Since X is semiregular, there is a regular open basis B and we have a regular open cover $\{B_i^j |$ for each $i\in I$, $O_i=\cup_j B_i^j$ where $B_i^j\in B\}$. By hypothesis, there exists a finite subfamily $\{B_{i_k}^j\}$ of $\{B_i^j\}$ such that $X=\cup_k \mathrm{Int}(B_{i_k}^j)$. Hence we have a finite subcover of $\{O_i\}$, thus X is compact.

REMARK [5]. *A T_1-space X is metrizable if and only if it is semiregular.*

THEOREM 2.2. *Each RS-compact metrizable space is finite.*

Proof. Suppose that an RS-compact metrizable space X is infinite. Since X is compact by Lemma 2.1, it is not discrete. Thus X has an accumulation point x. Let $\{U_n | n\in N\}$ be a local base at x such that $U_1=X$, U_n is open in X and $\mathrm{Cl}(U_{n+1})\subset U_n$ for each $n\in N$. Let $\{N_k | k\in N\}$ be a family of pairwise disjoint infinite subsets of N (N=the set of positive integers) such that $\cup\{N_k | k\in N\}=N$. For each $k\in N$, we set $V_k=\{x\}\cup\cup\{Cl(U_n)-\mathrm{Cl}(U_{n+1}) | n\in N_k\}$. Then $\{V_k | k\in N\}$ is a regular semi-open cover of X. If $n\in N$, then $\cup\{\mathrm{Int}(V_k) | k\leq n\}\subset\cup\{\mathrm{Cl}(V_k) | k\leq n\}\neq X$. Thus X is not RS-compact. This contradicts.

3. Irresolute images of weak *RS*-compact spaces

In this section we shall show that the irresolute image of any weak RS-compact Hausdorff space in any Hausdorff space is closed.

DEFINITION 3.1. A filterbase $F=\{A_i\}$ is said to be rs-accumulate a point x if for every regular semi-open set V containing x and for every $A_i\in F$, $A_i\cap\mathrm{Cl}(V)\neq\phi$.

THEOREM 3.2. *For a topological space the following are equivalent:*

a) *X is weak RS-compact.*

b) *For each family of regular semi-closed* (i.e., *the complement of a regular semi-open set is regular semi-closed.*) *sets $\{F_\alpha\}$ such that $\cap F_\alpha=\phi$, then there exists a finite subfamily $\{F_{\alpha_i}\}$ such that $\cap_{i=1}^n F_{\alpha_i}=\phi$.*

c) *Each filterbase $F=\{A_i\}$ rs-accumulates to some point $x\in X$.*

Proof. a)$\Longrightarrow$c). Let $F=\{A_i\}$ be a filterbase. Suppose that F does not *rs*-accumulate to any point. This implies that every $x\in X$, there exists a regular semi-open set V_x containing x and an $A_{ix}\in F$ such that $A_{ix}\cap \mathrm{Cl}(V_x)=\phi$. Obviously $\{V_x | x\in X\}$ is a regular semi-open cover of X and by hypothesis there exists a finite subfamily such that $\cup_{i=1}^n V_{x_i}=X(=\cap_{i=1}^n \mathrm{Cl}(V_{x_i}))$. Since F is filterbase, there exists an $A_j\in F$ such that $A_j\subset \cap_{i=1}^n A_{ix_i}$. Hence, $A_j\cap \mathrm{Cl}(V_{x_i})=\phi$, for each $i=1,2,\dots,n$, which implies $A_j\cap(\cup_{i=1}^n \mathrm{Cl}(V_{x_i}))=A_j\cap X=\phi$, contradicting the fact that $A_j\neq\phi$.

c)$\Longrightarrow$b). Let $\{F_\alpha\}$ be a collection of regular semi-closed sets such that $\cap F_\alpha=\phi$. Suppose that for every finite subfamily, $\cap_{i=1}^n F_{\alpha_i}\neq\phi$. Therefore $F=\{\cap_{i=1}^n F_{\alpha_i} | n\in N,\ F_{\alpha_i}\in\{F_\alpha\}\}$ forms a filterbase. By hypothesis, F *rs*-accumulates to some point $x\in X$. This implies that for every regular semi-open set V_x containing x, $F_\alpha\cap Cl(V_x)\neq\phi$, for every $\alpha\in I$. Since $x\notin\cap F_\alpha$ there exists an $j\in I$ such that $x\notin F_j$. Hence x is contained in the regular semi-open set $X-F_j$. Therefore

$$F_j\cap Cl(X-F_j)\subseteq F_j\cap(X-\mathrm{Int}(F_j))=\phi,$$

contradicting the fact that F *rs*-accumulates to the point x.

b)$\Longrightarrow$a) The proof is omitted (cf. Theorem 2.2 [3]).

THEOREM 3.3. *The irresolute image of any weak RS-compact Hausdorff space in any Hausdorff space is closed.*

Proof. Let $f: X\rightarrow Y$ be an irresolute function from a weak *RS*-compact Hausdorff space X to a Hausdorff space Y. Let $y\in \mathrm{Cl}(f(X))$ and $N(y)$ be the open neighborhood filterbase about y. By hypothesis, the filterbase $F=f^{-1}\{N(y)\}$ has an *rs*-accumulation point x. We claim that the filterbase $f(F)$ accumulates to $f(x)$ in the usual sense. Indeed, let V be any open set containing $f(x)$. Then $f^{-1}(V)$ is a semi-open set containing x, and therefore for every $W\in N(y)$, $f^{-1}(W)\in F$, and $f^{-1}(W)\cap \mathrm{Cl}(f^{-1}(V))\neq\phi$. We have $\mathrm{Int}(f^{-1}(W))\cap \mathrm{Int}(f^{-1}(V))\neq\phi$. Therefore, $W\cap V\supset f(f^{-1}(W)\cap f^{-1}(V))\supset f(\mathrm{Int}(f^{-1}(W))\cap \mathrm{Int}(f^{-1}(V)))\neq\phi$. Since W and V were arbitraily chosen, we have that $f(F)$ accumulates to $f(x)$ in the usual sense. But $f(F)$ is a finer filterbase than $N(y)$, hence $N(y)$ accumulates to $f(x)$. Since

$N(y)$ obviously converses to y, we have by the Hausdorff property that $f(x)=y$. Hence, $y \in f(X)$ and $f(X)$ is closed in Y.

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