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THE UNIT GROUP OF THE INTEGRAL GROUP RING $\mathbf{Z}D_8$

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1. Introduction

The purpose of this papers is to determine the unit group $U(\mathbf{Z}D_8)$, where D_8 is the dihedral group of order 8. In fact, the unit group $U(\mathbf{Z}D_8)$ has been studied in [1] and [5]. We will determine $U(\mathbf{Z}D_8)$ by the different method. The following is the main theorem in this paper.

THEOREM *Let*

$$D_8=\langle x, y : x^4=1,\ y^2=1,\ y^{-1}xy=x^{-1}\rangle$$

be the dihedral group of order 8. *Then* $U(\mathbf{Z}D_8)$ *is isomorphic to*

$$\left\{\left(e,\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)\in\mathbf{Z}\oplus M_2(\mathbf{Z}) : \begin{matrix} e=\pm1,\ \delta=1,\ a\equiv d\equiv\pm1 \pmod 4\\ b\equiv c\equiv 0 \text{ or } 2 \pmod 4\end{matrix}\right\}$$
$$\cup\left\{\left(e,\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)\in\mathbf{Z}\oplus M_2(\mathbf{Z}) : \begin{matrix} e=\pm1,\ \delta=1,\ b\equiv -c\equiv\pm1 \pmod 4\\ a\equiv d\equiv 0 \text{ or } 2 \pmod 4\end{matrix}\right\}$$
$$\cup\left\{\left(e,\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)\in\mathbf{Z}\oplus M_2(\mathbf{Z}) : \begin{matrix} e=\pm1,\ \delta=-1, a\equiv -d\equiv\pm1 \pmod 4\\ b\equiv c\equiv 0 \text{ or } 2 \pmod 4\end{matrix}\right\}$$
$$\cup\left\{\left(e,\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)\in\mathbf{Z}\oplus M_2(\mathbf{Z}) : \begin{matrix} e=\pm1,\ \delta=-1, b\equiv c\equiv\pm1 \pmod 4\\ a\equiv d\equiv 0 \text{ or } 2 \pmod 4\end{matrix}\right\},$$

where $\delta=ad-bc$.

The notations in this paper are standard. In particular, we will denote by $M_n(R)$ the full matrix ring of degree n over a commutative ring R with 1. Thus

$$U(M_n(R))=\{A\in M_n(R) : \det A\in U(R)\}.$$

Since $U(\mathbf{Z})=\{1,-1\}$, it follows that

$$U(M_n(\mathbf{Z}))=\{A\in M_n(\mathbf{Z}) : \det A=\pm1\}.$$

2. The proof of Theorem

In this section we will prove our main theorem.

First of all, we will show that the group algebra $\mathbf{Q}D_8$ over the rational field $\mathbf{Q}$ is isomorphic to $\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus M_2(\mathbf{Q})$.

Consider the map

$$\theta : \mathbf{Q}D_8 \to \mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus M_2(\mathbf{Q})$$

given by

$$\theta(x) = (1, -1, 1, -1, X), \quad \theta(y) = (1, 1, -1, -1, Y)$$

where

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since $X^4 = 1$, $Y^2 = 1$ and $Y^{-1}XY = X^{-1}$, we have a well defined homomorphism by linear extension. In fact, if $(a_1, \ldots, a_8)$ denotes the element

$$a_1 + a_2x + a_3x^2 + a_4x^3 + a_5y + a_6xy + a_7x^2y + a_8x^3y$$

of $\mathbf{Q}D_8$ and if $(x_1, \ldots, x_8)$ denotes the element

$$\left(x_1, x_2, x_3, x_4, \begin{bmatrix} x_5 & x_6 \\ x_7 & x_8 \end{bmatrix}\right)$$

of $\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus M_2(\mathbf{Q})$, and we think of $(a_1, \ldots, a_8)$ and $(x_1, \ldots, x_8)$ as row vectors, then we have

$$(x_1, \ldots, x_8) = \theta(a_1, \ldots, a_8) = (a_1, \ldots, a_8)A,$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 \\ 1 & -1 & -1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 & -1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad A^{-1} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 & -2 & 0 & 2 \\ 0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 & -2 & 0 & 2 & 0 \end{pmatrix}$$

Since A^{-1} is an element in $M_8(\mathbf{Q})$, it follows that θ is an isomorphism of $\mathbf{Q}D_8$ onto $\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus M_2(\mathbf{Q})$.

Now it is clear that

$$\mathbf{Z}D_8 \cong \theta(\mathbf{Z}D_8) \subseteq \mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}\oplus M_2(\mathbf{Z}).$$

An element $(x_1, \ldots, x_8)$ of $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}\oplus M_2(\mathbf{Z})$ is in $\theta(\mathbf{Z}D_8)$ if and only if $(a_1, \ldots, a_8) = (x_1, \ldots, x_8)A^{-1} \in \mathbf{Z}D_8$. Thus $(x_1, \ldots, x_8) \in \theta(\mathbf{Z}D_8)$ if and only if $a_1, \ldots, a_8 \in \mathbf{Z}$, where

$$
(2.1)\quad
\begin{aligned}
a_1&=\frac{1}{8}(x_1+x_2+x_3+x_4+2x_5+2x_8)\\
a_2&=\frac{1}{8}(x_1-x_2+x_3-x_4+2x_6-2x_7)\\
a_3&=\frac{1}{8}(x_1+x_2+x_3+x_4-2x_5-2x_8)\\
a_4&=\frac{1}{8}(x_1-x_2+x_3-x_4-2x_6+2x_7)\\
a_5&=\frac{1}{8}(x_1+x_2-x_3-x_4+2x_5-2x_8)\\
a_6&=\frac{1}{8}(x_1-x_2-x_3+x_4-2x_6-2x_7)\\
a_7&=\frac{1}{8}(x_1+x_2-x_3-x_4-2x_5+2x_8)\\
a_8&=\frac{1}{8}(x_1-x_2-x_3+x_4+2x_6+2x_7)
\end{aligned}
$$

Therefore, $(x_1, \ldots, x_8)\in\theta(\mathbf{Z}D_8)$ if and only if $(x_1, \ldots, x_8)$ satisfies the following system of congruences:

$$
(2.2)\quad
\begin{aligned}
&x_1+x_2+x_3+x_4+2x_5+2x_8\equiv 0 \pmod 8\\
&x_1-x_2+x_3-x_4+2x_6-2x_7\equiv 0 \pmod 8\\
&x_1+x_2+x_3+x_4-2x_5-2x_8\equiv 0 \pmod 8\\
&x_1-x_2+x_3-x_4-2x_6+2x_7\equiv 0 \pmod 8\\
&x_1+x_2-x_3-x_4+2x_5-2x_8\equiv 0 \pmod 8\\
&x_1-x_2-x_3+x_4-2x_6-2x_7\equiv 0 \pmod 8\\
&x_1+x_2-x_3-x_4-2x_5+2x_8\equiv 0 \pmod 8\\
&x_1-x_2-x_3+x_4+2x_6+2x_7\equiv 0 \pmod 8
\end{aligned}
$$

It is easy to see by row reduction that the above system is equivalent to the following system of congruences:

$$
(2.3)\quad
\begin{aligned}
&x_5\equiv x_8 \pmod 2\\
&x_6\equiv x_7 \pmod 2\\
&x_4\equiv x_5+x_6 \pmod 2\\
&x_3+x_4\equiv 2x_5 \pmod 4\\
&x_2+x_4\equiv x_5+x_6-x_7+x_8 \pmod 4\\
&x_1+x_2+x_3+x_4\equiv 2x_5+2x_8 \pmod 8
\end{aligned}
$$

An element $(x_1, \ldots, x_8)$ of $\theta(\mathbf{Z}D_8)$ is a unit of $\theta(\mathbf{Z}D_8)$ if and only if the inverse $(x_1, \ldots, x_8)^{-1}$ exists and is in $\theta(\mathbf{Z}D_8)$. Hence $(x_1, \ldots, x_8)$ is a unit of $\theta(\mathbf{Z}D_8)$ if and only if the following hold:

$$x_1=\pm1,\ x_2=\pm1,\ x_3=\pm1,\ x_4=\pm1,\ \delta=x_5x_8-x_6x_7=\pm1,$$

$(x_1, \ldots, x_8)$ satisfies (2.3), and

$(x_1, \ldots, x_8)^{-1}=(x_1, x_2, x_3, x_4, \delta x_8-\delta x_6, -\delta x_7, \delta x_5)$ satisfies (2.3).

Note that $x_5\equiv x_8 \pmod 2$ is equivalent to

$$x_5+x_8\equiv0 \pmod 4 \text{ or } x_5+x_8\equiv2 \pmod 4$$

and $\pm1=x_4\equiv x_5+x_6 \pmod 2$ is equivalent to $x_5\not\equiv x_6 \pmod 2$. Therefore, $(x_1, \ldots, x_8)$ is a unit of $\theta(\mathbf{Z}D_8)$ if and only if $(x_1, \ldots, x_8)$ satisfies the following:

$$x_1=\pm1, x_2=\pm1,\ x_3=\pm1,\ x_4=\pm1,\ \delta=x_5x_8-x_6x_7=\pm1,$$

$$x_5+x_8\equiv0 \pmod 4 \textit{ or } x_5+x_8\equiv2 \pmod 4,$$

$$x_6\equiv x_7 \pmod 2,$$

$$(2.4)\quad x_5\not\equiv x_6 \pmod 2,$$

$$x_3+x_4\equiv2x_5\equiv2\delta x_8 \pmod 4,$$

$$x_2+x_4\equiv x_5+x_8+x_6-x_7\equiv\delta(x_5+x_8)-\delta(x_6-x_7) \pmod 4,$$

$$x_1+x_2+x_3+x_4\equiv2(x_5+x_8)\equiv2\delta(x_5+x_8) \pmod 8.$$

Finally we determine the unit group $U(\theta\mathbf{Z}D_8)$ of the ring $\theta(\mathbf{Z}D_8)$. In fact, we will show that $U(\theta\mathbf{Z}D_8)$ is the set of all $(x_1, \ldots, x_8)$ satisfying one of the following:

(2.5)

(i) $x_1=x_2=x_3=x_4=\pm1,\ \delta=1$

$x_5\equiv x_8\equiv1$ *or* $-1 \pmod 4$, $x_6\equiv x_7\equiv0$ *or* $2 \pmod 4$

(ii) $x_1=-x_2=x_3=-x_4=\pm1,\ \delta=1$

$x_5\equiv x_8\equiv0$ *or* $2 \pmod 4$, $x_6\equiv -x_7\equiv1$ *or* $-1 \pmod 4$

(iii) $x_1=x_2=-x_3=-x_4=\pm1,\ \delta=-1$

$x_5\equiv -x_8\equiv1$ *or* $-1 \pmod 4$, $x_6\equiv x_7\equiv0$ *or* $2 \pmod 4$

(iv) $x_1=-x_2=-x_3=x_4=\pm1,\ \delta=-1$

$x_5\equiv x_8\equiv0$ *or* $2 \pmod 4$, $x_6\equiv x_7\equiv1$ or $-1 \pmod 4$

In order to prove this fact, let $(x_1, \ldots, x_8)$ be a unit of $\theta(\mathbf{Z}D_8)$. Then we have either $x_5+x_8\equiv 0 \pmod 4$ or $x_5+x_8\equiv 2 \pmod 4$.

First consider the case when $x_5+x_8\equiv 2 \pmod 4$. It follows from (2.4) that $x_1+x_2+x_3+x_4\equiv 4 \pmod 8$ and $x_2+x_4\equiv 2+(x_6-x_7) \pmod 4$. Hence

$$x_1=x_2=x_3=x_4=\pm 1,\quad 2\equiv 2x_2\equiv 2+(x_6-x_7) \pmod 4,$$
$$2\equiv 2x_3\equiv 2x_5 \pmod 4.$$

Thus $x_6\equiv x_7 \pmod 4$, $x_5\equiv 1 \pmod 2$ and so $x_6\equiv 0 \pmod 2$. By the above results it is easy to see that

$$x_5\equiv x_8\equiv 1 \text{ or } -1 \pmod 4,$$
$$x_6\equiv x_7\equiv 0 \text{ or } 2 \pmod 4.$$

Moreover, $\delta=x_5x_8-x_6x_7\equiv 1 \pmod 4$ and so $\delta=1$. Hence the case (i) holds.

Next consider the case when $x_5+x_8\equiv 0 \pmod 4$. It follows from (2.4) that $x_1+x_2+x_3+x_4\equiv 0 \pmod 8$ and $x_2+x_4\equiv x_6-x_7 \pmod 4$. Hence we have $x_1+x_2+x_3+x_4=0$. Note that $x_2=x_4$ or $x_2+x_4=0$. If $x_2=x_4$, then $2\equiv 2x_2\equiv x_6-x_7 \pmod 4$ and $x_1+x_3\equiv -2x_2\equiv 2 \pmod 4$. Therefore, $x_1=x_3$ and so $x_1=-x_2=x_3=-x_4=\pm 1$. Thus $0\equiv 2x_5 \pmod 4$ and so $x_5\equiv 0 \pmod 2$ and $x_6\equiv 1 \pmod 2$. By the above results it is easy to see that

$$x_5\equiv x_8\equiv 0 \text{ or } 2 \pmod 4,$$
$$x_6\equiv -x_7\equiv 1 \text{ or } -1 \pmod 4.$$

Moreover, $\delta=x_5x_8-x_6x_7\equiv 1 \pmod 4$ and so $\delta=1$. Hence the case (ii) holds. Now assume that $x_2+x_4=0$. Then $x_1+x_3=0$ and $x_6-x_7\equiv 0 \pmod 4$. Hence we have $x_1=x_2=-x_3=-x_4=\pm 1$ or $x_1=-x_2=-x_3=x_4=\pm 1$. If $x_1=x_2=-x_3=-x_4=\pm 1$, then $2\equiv 2x_3\equiv 2x_5 \pmod 4$ and so $x_5\equiv 1 \pmod 2$, $x_6\equiv 0 \pmod 2$. Thus we have

$$x_5\equiv -x_8\equiv 1 \text{ or } -1 \pmod 4,$$
$$x_6\equiv x_7\equiv 0 \text{ or } 2 \pmod 4,$$
$$\delta=-1.$$

Hence the case (iii) holds. On the other hand, if $x_1=-x_2=-x_3=x_4=\pm 1$, then $0\equiv 2x_5 \pmod 4$ and so $x_5\equiv 0 \pmod 2$, $x_6\equiv 1 \pmod 2$. Thus we have

$$x_5\equiv x_8\equiv 0 \text{ or } 2 \pmod 4,$$
$$x_6\equiv x_7\equiv 1 \text{ or } -1 \pmod 4,$$
$$\delta=-1.$$

Therefore, the case (iv) holds and all.

Conversely, it is not hard to show that if an element $(x_1, \ldots, x_8)$ of $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}\oplus M_2(\mathbf{Z})$ satisfies any one of the conditions (i)$\sim$(iv) of (2.5) then it satisfies (2.4) and so it is a unit of $\theta(\mathbf{Z}D_8)$.

The group homomorphism of $U(\theta\mathbf{Z}D_8)$ into $U(\mathbf{Z}\oplus M_2(\mathbf{Z}))$ defined by

$$(x_1, x_2, x_3, x_4, \begin{bmatrix} x_5 & x_6 \\ x_7 & x_8 \end{bmatrix}) \longrightarrow (x_4, \begin{bmatrix} x_5 & x_6 \\ x_7 & x_8 \end{bmatrix})$$

is a monomorphism by (2.5). Moreover, the image of the map can be found from (2.5). Since $U(\mathbf{Z}D_8)\cong U(\theta\mathbf{Z}D_8)$, this completes the proof of Theorem.

REMARK. Using (2.1) and (2.5), we can obtain the following result. This has been proved in [1]:

The element

$$a_1+a_2x+a_3x^2+a_4x^3+a_5y+a_6xy+a_7x^2y+a_8x^3y$$

of $\mathbf{Z}D_8$ is a unit if and only if one of the following holds.

(1) $a_1+a_3=\pm1,\ a_2+a_4=0,\ a_5+a_7=0,\ a_6+a_8=0,$
$(a_1-a_3)^2-4a_5^2-4a_6^2+4a_2^2=1$

(2) $a_2+a_4=\pm1,\ a_1+a_3=0,\ a_5+a_7=0,\ a_6+a_8=0,$
$4a_1^2-4a_5^2-4a_6^2+(a_2-a_4)^2=1$

(3) $a_5+a_7=\pm1,\ a_1+a_3=0,\ a_2+a_4=0,\ a_6+a_8=0,$
$-4a_1^2+(a_5-a_7)^2+4a_6^2-4a_2^2=1$

(4) $a_6+a_8=\pm1,\ a_1+a_3=0,\ a_2+a_4=0,\ a_5+a_7=0,$
$-4a_1^2+4a_5^2+(a_6-a_8)^2-4a_2^2=1$

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