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# ON THE LOCAL CONVERGENCE IN MEASURE

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Let $(X, \mathfrak{B}, \mu)$ be a measure space. Let $\mathfrak{M}$ denote the set of all real valued measurable functions on $X$, where we have identified functions which are equal $\mu$-almost everywhere. It is well known that if $\mu$ is finite, $\mathfrak{M}$ is a complete metric space with the metric

$$d(f, g)=\int_X \frac{|f-g|}{1+|f-g|}d\mu, \quad f, g\in\mathfrak{M}$$

which is compatible with convergence in measure. Indeed, this fact follows from the following known result [1].

THEOREM 1. *Let $<f_n>$ be a sequence of real valued measurable functions on $(X, \mathfrak{B}, \mu)$ which is Cauchy in measure. Then there exists a subsequence which converges almost everywhere and in measure to a real valued measurable function $f$ on $X$.*

In this note we shall show that if $\mu$ is $\sigma$-finite, Theorem 1 can be extended (see Theorem 2) to apply to sequences of real valued measurable functions which are Cauchy in weaker type of convergence than convergence in measure. Now we begin with a definition.

DEFINITION. Let $f, f_1, f_2, \ldots$ be real valued measurable functions on $(X, \mathfrak{B}, \mu)$. The sequence $<f_n>$ is said to converge locally in measure to $f$ if for each $\alpha>0$, and $E\in\mathfrak{B}$ with $\mu(E)<\infty$

$$\lim_{n\to\infty}\mu(\{x\in E : |f_n(x)-f(x)|\geq\alpha\})=0.$$

The sequence $<f_n>$ is said to be Cauchy locally in measure if for each $\alpha>0$, and $E\in\mathfrak{B}$ with $\mu(E)<\infty$

$$\lim_{m,n\to\infty}\mu(\{x\in E : |f_m(x)-f_n(x)|\geq\alpha\})=0.$$

Obviously, if $<f_n>$ converges in measure to $f$, then $<f_n>$ converges locally in measure to $f$, but the converse need not be valid. For example, with $X=[0, \infty]$ and Lebesgue measure $\mu$, the sequence

$$f_n(x)=\begin{cases}1, & n<x\leq n+1\\ 0, & \text{otherwise}\end{cases}$$

converges to zero locally in measure but fails to converge to zero in measure. Thus local convergence in measure is weaker than convergence in measure.

THEOREM 2. *Let* $(X, \mathfrak{B}, \mu)$ *be a* $\sigma$*-finite measure space and* $\langle f_n \rangle$ *a sequence of real valued measurable functions on* $X$ *which is Cauchy locally in measure. Then there exists a subsequence which converges almost everywhere and locally in measure to a real valued measurable function* $f$ *on* $X$.

*Proof.* Since $\mu$ is $\sigma$-finite, there exists a sequence $\langle E_n \rangle$ of disjoint subsets of $X$ in $\mathfrak{B}$ such that $\mu(E_n) < \infty$ and $\bigcup_{n=1}^{\infty} E_n = X$. Denote by $\mu_m$ the restriction of $\mu$ to $E_m$, by $\mathfrak{B}(E_m)$ the $\sigma$-algebra $\{B \cap E_m : B \in \mathfrak{B}\}$, and by $f_n^{(m)}$ tne rectriction of $f_n$ to $E_m$. Since the sequence $\langle f_n^{(1)} \rangle$ is a Cauchy sequence of real valued measurable functions on $(E_1, \mathfrak{B}(E_1), \mu_1)$, it follows from Theorem 1 that there exists a subsequence $\langle f_{1,k}^{(1)} \rangle$ of $\langle f_n^{(1)} \rangle$ which converges $\mu_1$-almost everywhere. Thus we have $N_1 \in \mathfrak{B}(E_1)$ such that $\mu(N_1) = 0$ and $\langle f_{1,k}^{(1)}(x) \rangle$ converges for each $x \in E_1 - N_1$. Since th esequence $\langle f_{1,k} \rangle$ is a subsequence of $\langle f_n \rangle$, by the same argument as above, we can choose a subsequence $\langle f_{2,k} \rangle$ of $\langle f_{1,k} \rangle$ such that $\langle f_{2,k}^{(2)}(x) \rangle$ converges $\mu_2$-almost everywhere. Thus we have $N_2 \in \mathfrak{B}(E_2)$ such that $\mu(N_2) = 0$ and $\langle f_{2,k}^{(2)}(x) \rangle$ converges for each $x \in E_2 - N_2$. If we continue this process, for each $j \geq 1$, we will obtain a subsequence $\langle f_{j,k} \rangle$ of $\langle f_n \rangle$ such that $\langle f_{j,k}^{(j)} \rangle$ converges on $(E_j, \mathfrak{B}(E_j), \mu_j)$ off a $\mu_j$ null set $N_j$. Furthermore, $\langle f_{j,k} \rangle$ is a subsequence of $\langle f_{j-1,k} \rangle$, where $f_{0,k} = f_k$ for all $k$. Now let $N = \bigcup_{j=1}^{\infty} N_j$, then $\mu(N) = \sum_{j=1}^{\infty} \mu_j(N_j) = 0$ since $E_j$'s are disjoint. Hence the subsequence $\langle f_{k,k} \rangle$ of $\langle f_n \rangle$ converges on $X$ off the $\mu$ null set $N$. If we define $f$ by

$$f(x) = \begin{cases} f_{k,k}(x), & x \notin N \\ 0, & x \in N \end{cases}$$

then the subsequence $\langle f_{k,k} \rangle$ converges almost everywhere to the real valued measurable function $f$ on $X$

To see that $\langle f_{k,k} \rangle$ converges locally in measure to $f$, for each $n$ let $f^{(n)}$ denote the restriction of $f$ to $E_n$. Since $\langle f_{k,k}^{(n)} \rangle$ converges to $f^{(n)}$ almost everywhere on the finite measure space $(E_n, \mathfrak{B}(E_n), \mu_n)$ it follows from Egoroff's theorem ([1], p. 74) that $\langle f_{k,k}^{(n)} \rangle$ converges to $f^{(n)}$ in measure on $E_n$. Hence $\langle f_{k,k} \rangle$ convreges locally in measure to $f$.

COROLLARY 3. *Let* $(X, \mathfrak{B}, \mu)$ *and* $\langle f_n \rangle$ *be as in Theorem 2. Then there* *exists a real valued measurable function on X to which the sequence converges locally in measure. This limit function f is uniquely determined almost everywhere.*

*Proof.* We have seen that there exists a subsequence $\langle f_{k,k}\rangle$ of $\langle f_n\rangle$ which converges locally in measure to a function $f$. Now it is enough to show that the original sequence converges locally in measure to $f$. However, this can be done by retracing the proof of Corollary 7.7 [1] with $E_n$ in place of $X$, where $E_n$ is as in the proof of Theorem 2.

Let $(X, \mathfrak{B}, \mu)$ be a $\sigma$-finite measure space and $\langle B_n\rangle$ an increasing sequence of subsets of $X$ in $\mathfrak{B}$ with $\mu(B_n)<\infty$. Define a metric $d_n$ on $\mathfrak{M}$ by

$$d_n(f, g)=\int_{B_n}\frac{|f-g|}{1+|f-g|}d\mu, \quad f, g\in\mathfrak{M}.$$

Then, it is easy to see that local convergence in measure is compatible with the metric on $\mathfrak{M}$

$$d(f, g)=\sum_{n=1}^{\infty}\frac{2^{-n}d_n(f, g)}{1+d_n(f, g)}.$$

Thus, Corollary 3 shows that $m$ becomes a complete metric space with the metric $d$.

The following corollary can be easily shown by using Theorem2 and standard arguments (see [1]).

COROLLARY 4. *Let $(X, \mathfrak{B}, \mu)$ be a $\sigma$-finite measure space. Then Fatou's Lemma and the Monotone and Lebesgue Dominate Convergence theorem on X remain valid if convergence in measure is replaced by local convergence in measure.*

## Reference

1. R.G. Bartle, *The Elements of Integration*, Wiley, New York, 1966.

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