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ON THE SUBMARTINGALE CONVERGENCE IN BURKHOLDER TRANSFORMS

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1. Introduction

Let (Ω, F, P) be a probability space and $(F_n, n\in\mathbf{N})$ an increasing sequence of sub-σ-algebras of F. Let $X=(X_n, F_n, n\in\mathbf{N})$ be a submartingale on (Ω, F, P), and let $v=(v_n, n\in\mathbf{N})$ be a *predictable* sequence: $v_k : \Omega\to\mathbf{R}$ is F_{k-1}-measurable, $k\geq 1$, and $F_0=F_1$.

DEFINITION 1.1. The transform of X, $T(X)=(T_1(X), T_2(X), \ldots)$, will be defined as $T_n(X)=\sum_{k=1}^{n} v_k x_k$ where $x_1=X_1$, $x_2=X_2-X_1, \ldots, X_n=X_n-X_{n-1}, \ldots$.

Burkholder [2] defined the concept of martingale transforms and studied its convergence. In this paper we obtain some results on the convergence of the transforms of an L^1-bounded submartingale.

Our first result (Theorem 2.2) is that the transform of any L^1-bounded positive increasing submartingale converges almost surely (a.s.) on $\{v^*<\infty\}$ where $v^*(\omega)=\sup_n|v_n(\omega)|$, $\omega\in\Omega$. This result implies that the transform of any L^1-bounded increasing process converges a.s. on $\{v^*<\infty\}$.

The second result (Theorem 2.3) is that the transform of any L^1-bounded submartingale converges a.s. on $\{v^*<\infty\}$. This result is surely a generalization of Theorem 1 of [2].

The third result (Theorem 2.6) is that for every stopping time T the stopped r.v. X_T is integrable for any L^1-bounded submartingale $(X_n, F_n, n\in\mathbf{N})$.

The final result (Theorem 2.8) is a generalization of Theorem 3 of [2]. This final result will be proved by using our result, Theorem 2.7 which is a generalization of a result of [1].

The following propostion will be used in the sequel. This proposition is known as the Doob's decomposition theorem (see details in [6] or in [7]).

PROPOSITION 1.2. *For any submartingale* $(X_n, F_n, n\in\mathbf{N})$, *there exist uniquely a martingale* $(Y_n, F_n, n\in\mathbf{N})$ *and an increasing process* $(A_n, F_n, n\in\mathbf{N})$ *such that, for each* $n\in\mathbf{N}$ $\quad X_n=Y_n+A_n$.

REMARK 1.3. If the submartingale $(X_n, n\in\mathbf{N})$ is L^1-bounded [uniformly integrable (u.i.), resp.] then both the martingale $(Y_n, n\in\mathbf{N})$ and the increasing process $(A_n, n\in\mathbf{N})$ are also L^1-bounded [u.i. resp]. In particular we remark that the increasing process can be regarded as an increasing positive submartingale.

2. Main results

LEMMA 2.1 *If g is a transform of a uniformly bounded submartingale f and $v^*\leq 1$, then g converges a.s.*

Proof. Refer to the step (ii) of Theorem 1 of [2].

THEOREM 2.2. *Let $(X_n, n\in\mathbf{N})$ be an L^1-bounded nonnegative increasing submartingale. Then the transform $(T_n(X), n\in\mathbf{N})$ converges a.s. on $\{v^*<\infty\}$.*

Proof. Let $v^*\leq 1$ and let $c>0$. Then $\hat{X}_n=\min(X_n, c)$ defines a uniformly bounded submartingale $(\hat{X}_n, n\in\mathbf{N})$. By Lemma 2.1, $T_n(X)$ converges a.s. But since $T_n(X)=T_n(\hat{X})$ if $X^*(\omega)\equiv\sup_n|X_n(\omega)|<c$, $\omega\in\Omega$, we have that $T_n(X)$ converges a.s. on the set $\{X^*<c\}$. Since $(X_n, n\in\mathbf{N})$ is L^1-bounded, it follows that $EX^*=E(\lim_n X_n)\leq\sup_n EX_n<\infty$. Hence we have $P(X^*<\infty)=1$. Therefore, letting $c\to\infty$ we have that

(2.1) $T_n(X)$ converges a.s. if $v^*\leq 1$.

Let $\hat{v}_n(\omega)=v_n(\omega)$ if $|v_n(\omega)|<c$, $\hat{v}_n(\omega)=0$ otherwise. Let $\hat{T}_n(X)$ be the transform of X under the uniformly bounded multiplier sequence $(\hat{v}_n, n\in\mathbf{N})$. Clearly, (2.1) implies that $\hat{T}_n(X)$ converges a.s. Since $T_n(X)=\hat{T}_n(X)$ if $v^*<c$, we have that $T_n(X)$ converges a.s. on $\{v^*<c\}$. Since c is arbitrary, it follows that $T_n(X)$ converges a.s. on $\{v^*<\infty)$.

Using Doob's decomposition, we obtain that the transform of any L^1-bounded submartingale converges a.s. on $\{v^*<\infty\}$. This theorem is a generalized result of Theorem 1 of [2].

THEOREM 2.3. *Let $(X_n, n\in\mathbf{N})$ be an L^1-bounded submartingale. Then the transform $(T_n(X), n\in\mathbf{N})$ converges a.s. on $\{v^*<\infty\}$.*

Proof. Let $X_n=Y_n+A_n$, $n\in\mathbf{N}$, be the Doob's decomposition of $(X_n, n\in\mathbf{N})$. Then we have $T_n(X)=T_n(Y)+T_n(A)$. Therefore, by Theorem 1 of [2], Theorem 2.2 and Remark 1.3 we obtain the desired result.

COROLLARY 2.4 *Let $(X_n, n\in\mathbf{N})$ be an L^1-bounded submartingale. If A is an atom of F, then $\sum_{n=1}^{\infty}|x_n|<\infty$ a.s. on A.*

Proof. There is a real number sequence $a=(a_1, a_2, ...)$ such that $x_n=a_n$ a.s. on A, otherwise there would be a subset B of A in F such that $0< P(B)<P(A)$, contradicting the assumption that A is an atom. Let $v_n(\omega)=1$ if $a_n\geq 0$, $=-1$ if $a_n<0$, $\omega\in\Omega$. By Theorem 2.3, $T_n(X)=\sum_{k=1}^{n} v_k x_k$ converges a.s. on A.

PROPOSITION 2.5. *Let $(X_n, n\in\mathbf{N})$ is a martingale or a nonnegative submartingale which is bounded in L^1. Then X_T is integrable for every stopping time T.*

The above proposition can be generalized as the following theorem.

THEOREM 2.6. *Let $(X_n, n\in\mathbf{N})$ be an L^1-bounded submartingale. Then for any stopping time T, X_T is integrable.*

Proof. Let $X_n=Y_n+A_n$ be the Doob's decomposition. By Proposition 2.5 and Remark 1.3, we have

$$E|X_T|\leqslant E|Y_T|+EA_\infty<\infty.$$

This completes the proof.

THEOREM 2.7. *Let $(X_n, F_n, n\in\mathbf{N})$ be an L^1-bounded submartingale. Then its square function $S(X)\equiv\lim_n (\sum_{k=1}^{n} x_k^2)^{\frac{1}{2}}$ is finite a.s.*

Proof. By Proposition 1.2 and Remark 1.3, there exist an L^1-bounded martingale $(Y_n, F_n, n\in\mathbf{N})$ and a uniformly integrable increasing process $(A_n, F_n, n\in\mathbf{N})$ such that $X_n=Y_n+A_n$ and $\lim_n A_n=A_\infty$ is integrable.

Since for any $a>0$, $b>0$, $(a+b)^2\leqslant 2^2(a^2+b^2)$, we obtain, for each $n\in\mathbf{N}$

$$S_n^2(X)\equiv\sum_{k=1}^{n} x_k^2\leqslant 4(S_n^2(Y)+S_n^2(A)). \tag{2.2}$$

Since for any $a>0, b>0$, $(a+b)^{\frac{1}{2}}\leqslant a^{\frac{1}{2}}+b^{\frac{1}{2}}$, it follows from (2.2) that for each n, we have

$$S_n(X)\leqslant 2(S_n(Y)+S_n(A)). \tag{2.3}$$

From the nonnegativity of $S_n(A)$ and A_n, and since $S_n^2(A)\leqslant A_n^2$, we obtain, for each $n\in\mathbf{N}$

$$S_n(A)\leqslant A_n \tag{2.4}$$

Taking the supremum on the both sides of (2.3) and (2.4), we obtain that $S(X)\leqslant 2(S(Y)+S(A))$ and $S(A)\leqslant A_\infty$ which imply

$$S(X)\leqslant 2(S(Y)+A_\infty). \tag{2.5}$$

By Austin [1] and the integrability of A_∞ and (2.5), we obtain $P(S(X)<\infty)=1$. The proof is completed.

The following theorem is a generalized result of Theorem 3 of [2]. To prove this theorem we will use Theorem 2.7.

THEOREM 2.8. *Let* $X=(X_n, F_n, n\in \mathbf{N})$ *be an* L^1*-bounded submartingale and* $Y=(Y_n, F_n, n\in \mathbf{N})$ *a martingale. If* $S_n(Y)\leqslant S_n(X)$, $n\in \mathbf{N}$, *then* Y_n *converges a. s.*

Proof. Let $c>0$, and let $m(\omega)=\inf\{n : |X_n(\omega)|\geq c$ or $S_n(X(\omega))\geq c\}$ where $\inf\phi=\infty$. Then $ES_m(X)<\infty$. For

$$S_m(X)<c+|x_m|<2c+|X_m| \quad \text{on } \{m<\infty\},$$

$$S_m(X)\leqslant c \text{ on } \{m=\infty\}.$$

By Theorem 2.6, $E|X_m|<\infty$ which implies $ES_m(X)<\infty$.

Let $\hat{Y}_n=Y_{m\wedge n}$. Then $(\hat{Y}_n,\ n\in \mathbf{N})$ is a martingale by the Doob's optional stopping theorem (see details in [6]). Here we have used our assumption that X and Y are relative to the same sequence of sub-σ-algebras of F. Clearly, $S(\hat{Y})=S_m(Y)\leq S_m(X)$. Therefore, $ES(\hat{Y})<\infty$ and by Theorem 2 of [2], $\hat{Y}$ converges a. s.

On the set $\{X^*<c,\ S(X)<c\}$, we have $m=\infty$ and $Y=\hat{Y}$. Since X^* and $S(X)$ are finite a. s. by the L^1-boundedness of X and Theorem 2.7, it follows that Y converges a. s..

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