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ON A CLASS OF MALCEV-ADMISSIBLE ALGEBRAS

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1. Introduction

For an algebra A over a field F, denote by A^- the algebra with multiplication $[x, y]=xy-yx$ defined on the same vector space as A, where juxtaposition xy is the product in A. Then A is termed *Malcev-admissible* if A^- is a Malcev algebra; that is, the Malcev identity

$$[[x, y], [x, z]]=[[[x, y], z], x]+[[[y, z], x], x]+[[[z, x], x], y]$$

is satisfied for all $x, y, z\in A$. An algebra A is called *Lie-admissible* if A^- is a Lie algebra, namely, A^- satisfies the Jacobi identity

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$$

for $x, y, z\in A$. It is well known that every Lie-admissible and alternative algebra is Malcev-admissible [8]. Thus a Malcev-admissible algebra is a natural generalization of Lie-admissible, alternative, and Malcev algebras.

We assume hereafter that A is an algebra over F of characteristic $\neq 2$. Denote also by A^+ the algebra with multiplication $x\circ y=\frac{1}{2}(xy+yx)$ defined on the vector space A. The product xy in A is then given by

$$xy=\frac{1}{2}[x, y]+x\circ y. \tag{1}$$

Thus every Malcev algebra can occur as the attached algebra A^- for a Malcev-admissible algebra A by merely varying the commutative product $x\circ y$. This indicates that Malcev-admissibility alone is too broad a condition to yield a fruitful structure theory. It has been shown that the flexible law

$$(xy)x=x(yx) \tag{2}$$

for $x, y\in A$ is a useful restriction on A that imposes constraints on the product $x\circ y$ and its relations with the product $[\ ,\ ]$. It can be easily seen that the flexible law is equivalent to the fact that the adjoint mapping ad $x : y \rightarrow \mathrm{ad}x(y)=[x, y]$ is a derivation of A^+, i.e.,

$$[x, y\circ z]=y\circ[x, z] + [x, y]\circ z \tag{3}$$

for all $x, y, z\in A$.

The structure of finite-dimensional flexible Malcev-admissible algebras over a field F of characteristic 0 has recently been investigated by Myung [6] in

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light of the known structure for flexible Lie-admissible algebras. In this paper, we extend some of the results in [6] to the case of arbitrary characteristic $\neq 2$ and dimension, and also generalize those results by Benkart [1, 2] and Myung [5] for flexible Lie-admissible algebras to the Malcev case.

2. Flexible Malcev-admissible algebras

Let M be a Malcev algebra with product $[x, y]$ over a field F. A *Cartan subalgebra* H of M is defined as a nilpotent subalgebra of M such that the Fitting null component of M relative to ad H coincides with H; i.e.,

$$H=\{x\in M \mid (\mathrm{ad}\ h)^{n(h)}x=0,\ \ h\in H\}.$$

Let A be a flexible Malcev-admissible algebra with multiplication $x*y$ over F. Let H be a Cartan subalgebra of A^- for which there exists a Cartan decomposition

$$A^-=H\oplus \sum_{\alpha\neq 0}\oplus A_\alpha \tag{4}$$

where

$$A=\{x\in A \mid (\mathrm{ad}\ h-\alpha(h)I)^n x=0,\ \ h\in H\}$$

is the root space of A^- for H relative to the root $\alpha: H\to F$ (see [4], [9]). It can be shown [4] that

$$[A_\alpha, A_\beta]\subseteq A_{\alpha+\beta} \text{ for } \alpha\neq\beta, \tag{5}$$

$$[A_\alpha, A_\alpha]\subseteq A_{2\alpha}+A_{-\alpha}. \tag{6}$$

Note that, in a Lie algebra, (5) is valid for all roots α, β.

LEMMA 1. *Let A be a flexible Malcev-admissible algebra with product $x*y$ over F. Then every Cartan subalgebra H of A^- is a subalgebra of A. Furthermore, if A^- has a Cartan decomposition given by* (4) *then*

$$A_\alpha * A_\beta\subseteq A_{\alpha+\beta} \text{ for all } \alpha\neq\beta,$$
$$A_\alpha * A_\alpha\subseteq A_{2\alpha}+A_{-\alpha} \tag{7}$$

Proof. Since A is flexible, each ad h $(h\in H)$ is a derivation of A^+ and hence by a standard argument, we have $A_\alpha\circ A_\beta\subseteq A_{\alpha+\beta}$ for all α, β. In light of (5), (6) and (1), this gives (7). The same reasoning shows that any Cartan subalgebra of A^- is a subalgebra of A.

LEMMA 2. *Let A be a flexible Malcev-admissible algebra over a field F of characteristic $\neq 2$ and let H be an abelian Cartan subalgebra* ($[H, H]=0$) *of A^-. Assume that A^- has a Cartan decomposition relative to H given by* (4) *where each* ad h $(h\in H)$ *diagonally acts on A_α for all $\alpha\neq 0$;* i.e., $[h, x]=\alpha(h)x$ *for all $x\in A_\alpha$ and $h\in H$. Then*

(1) *If $h\in H$ and $x\in A_\alpha$ for $\alpha\neq 0$ then $h*x$ and $x*h$ are multiples of x.*

(2) *If* $x \in A_\alpha$, $y \in A_\beta$ *and* $\alpha \neq \pm\beta$ *for* $\alpha, \beta \neq 0$ *then* $x*y$ *is a multiple of* $[x, y]$.

(3) *If* $H*H=0$ *then* $x*y=\frac{1}{2}[x, y]$ *for all* $x, y \in A$ *and hence* A *is a Malcev algebra.*

(4) *If the center of* A^- *is zero and* H *is power-associative under the product "$*$" with the property that there exists a positive integer* n *such that* $h^n=0$ *for all* $h \in H$, *then* $H*H=0$ *and* A *is a Malcev algebra.*

Proof. We first note, under the assumption, each root α is a linear functional on H. Let α be a nonzero root of H. For $h, h' \in H$ and $x \in A_\alpha$, since ad x is a derivation of A^+, by (3) we have

$$\alpha(h \circ h')x=[h \circ h', x]=[h, x] \circ h'+h \circ [h', x]=\alpha(h)x \circ h'+\alpha(h')h \circ x. \tag{8}$$

First, we consider the special case where $h=h'$ and $h \notin \ker \alpha$, the kernel of α. Then $\alpha(h)^2 x=\alpha(h)[h, x]=\alpha(h)(h*x-x*h)$, while $\alpha(h)^2 x=\alpha(h)(h*h-h*x)$ from (8). This implies that $h*x$ is a multiple of $[h, x]$. If $h \circ h=0$ then $x \circ h=0$ by (8) and hence

$$h*x=\frac{1}{2}\alpha(h)x=\frac{1}{2}[h, x]. \tag{9}$$

Assume that $h \notin \ker \alpha$ and $h' \in \ker \alpha$. Thus, by (8), $\alpha(h \circ h')x=\alpha(h)x \circ h'$ and so $x \circ h'$ is a multiple of x. Since $[h', x]=h'*x-x*h'=0$, this implies that $h'*x$ and $x*h'$ are multiples of x. This proves part (1). In particular, if $h \circ h'=0$, by (8) $x \circ h'=0$ and so $h'*x=0=\frac{1}{2}\alpha(h')x=\frac{1}{2}[h', x]$. By the linearity of α, we conclude that if $H*H=0$ then (9) holds for all $h \in H$, $x \in A_\alpha$.

Suppose now that $x \in A_\alpha$ and $y \in A_\beta$ for $\alpha \neq 0$, $\beta \neq 0$ with $\alpha \neq \pm\beta$. It follows from (3) that

$$[h \circ x, y]=h \circ [x, y]+[h, y] \circ x=\beta(h)y \circ x+h \circ [x, y] \tag{10}$$

for $h \in H$. By part (1), the left side and the second term of right side are multiples of $[x, y]$. Since $\beta \neq 0$, we conclude that $x \circ y$ is a multiple of $[x, y]$ and so is $x*y$. This verifies part (2).

Assume $H*H=0$. For nonzero roots α, β, let $x \in A_\alpha$, $y \in A_\beta$. By (9), $h \circ x=0$ for all $h \in H$, $x \in A_\alpha$. This together with (10), (5) and (6) implies $\beta(h)x \circ y=0$ for all $h \in H$. Since $\beta \neq 0$, we have $x \circ y=0$ and hence

$$x*y=-y*x=\frac{1}{2}[x, y].$$

Therefore, it follows from this and (9) that A is a Malcev algebra.

For the proof of part (4), it suffices to show that $H*H=0$ under the hypotheses. The proof of this is based on the identity (3) and hence the same as that in [5] for flexible Lie-admissible algebras.

LEMMA 2 is a Malcev algebra analog of the result by Benkart [2] for flexible Lie-admissible algebras which also generalizes an earlier result of Myung [5] that has been proved under the added hypotheses of finite dimension and of dim $A_\alpha=1$ *for* $\alpha\neq 0$. As noted in [2], well known Lie algebras of infinite dimension or of prime characteristic satisfying the conditions in Lemma 2 are the Virasoro algebra or generalized Witt algebras.

A well known example for the non-Lie, Malcev case is the 7-dimensional simple split Malcev algebra. In fact, it is shown [4, 8, 9] that if M is a finite-dimensional simple, non-Lie, Malcev algebra over a field F of characteristic $\neq 2, 3$ such that M has a Cartan decomposition for a Cartan subalgebra H, then M is isomorphic to a 7-dimensional algebra C_0^- with basis $h, e_1, e_2, e_3, e_{-1}, e_{-2}, e_{-3}$ and with multiplication table given by

$$[h, e_i]=\alpha e_i,\ \ [h, e_{-i}]=-\alpha e_{-i},\ \ \alpha\neq 0 \text{ in } F,\ \ i=1,2,3,\ \ [e_{-i}, e_i]=h,$$
$$[e_i, e_j]=\alpha e_{-k},\ \ [e_{-i}, e_{-j}]=2e_k,\ \ (ijk)=(123), (231), (312), \tag{11}$$

and all other products are zero, where [,] is an anticommuntative product. Such algebra C_0^- is unique and is called the split simple (non-Lie) Malcev algebra. From the table (11), it is clear that C_0^- has the Cartan decomposition

$$C_0^-=H\oplus A_\alpha\oplus A_{-\alpha}$$

where $H=Fh$ is a one-dimensional Cartan subalgebra of C_0^- and

$$A_\alpha=\sum_{i=1}^{3} Fe_i,\ \ A_{-\alpha}=\sum_{i=1}^{3} Fe_{-i}$$

are the root spaces for roots $\alpha, -\alpha$.

THEOREM 3. *Let A be a flexible Malcev-admissible algebra over a field F of characteristic $\neq 2$ such that A^- is isomorphic to C_0^- given by* (11). *Then A is itself a Malcev algebra isomorphic to C_0^-.*

Prof. We identify A^- with C_0^- as an F-space and let A^- have the basis h, e_i, e_{-i} $(i=1,2,3)$ given by (11). Since $H=Fh$ is a subalgebra of A (Lemma 1), we can let

$$h*h=h\circ h=\lambda h,\ \ \lambda\in F.$$

In light of (3) and (11), one gets

$$[h\circ h, e_i]=h\circ[h, e_i]+[h, e_i]\circ h=2\alpha h\circ e_i=[h*h, e_i]=\lambda[h, e_i]=\lambda\alpha e_i.$$

Hence, by a similar argument applied to the e_{-i}, we have

$$h\circ e_i=\frac{1}{2}\lambda e_i \text{ for all } i\neq 0. \tag{12}$$

For $i, j=1,2,3$, we use(3)and (11) to get $[h, e_i\circ e_j]=e_i\circ[h, e_j]+[h, e_i]\circ e_j=2\alpha e_i\circ e_j$ and hence

$$e_i\circ e_j=e_{-i}\circ e_{-j}=0,\ \ i, j=1,2,3, \tag{13}$$

since $\pm 2\alpha$ is not a root of H.

From $[h, e_i \circ e_{-i}] = e_i \circ [h, e_{-i}] + [h, e_i] \circ e_{-i} = 0$, it follows that

$$e_i \circ e_{-i} = \mu_i h, \quad \mu_i \in F, \quad i = 1, 2, 3. \tag{14}$$

On the other hand, $[e_1, e_i \circ e_{-i}] = e_i \circ [e_1, e_{-i}] + [e_1, e_i] \circ e_{-i} = 0$ by (11) and (13). In view of (14), this gives $\mu_i = 0$, $i = 1, 2, 3$. Finally, from $[h \circ e_i, e_{-i}] = h \circ [e_i, e_{-i}] + [h, e_{-i}] \circ e_i = -h \circ h - \alpha e_{-i} \circ e_i = -\lambda h$ by (14) and from $[h \circ e_i, e_{-i}] = \frac{1}{2} \lambda [e_i, e_{-i}] = -\frac{1}{2} \lambda h$ by (12), we have $\lambda = \frac{1}{2} \lambda$ or $\lambda = 0$. In light of Lemma 2, this completes the proof.

Theorem 3 has been proved in [6] under the added assumption that F is algebraically closed and of characteristic 0.

3. Third power Malcev-admissible algebras

Theorem 3 or Lemma 2 does not provide a new simple flexible Malcev-admissible algebra which is not Malcev. However, there exists a non-Lie, flexible Lie-admissible algebra A such that A^- is a simple Lie algebra. Specifically, let A be a finite-dimensional flexible Lie-admissible algebra over an algebracially closed field F of characteristic 0 such that A^- is simple. It is shown [3, 7] that if A^- is not of type $A_n (n \geq 2)$ then A is a Lie algebra, and if A^- is of type $A_n (n \geq 2)$ then A is either Lie or isomorphic to an algebra with product $x*y$ given by

$$x*y = \frac{1}{2} [x, y] + \beta x \# y, \quad \beta \neq 0 \text{ in } F,$$

defined on the vector space $sl(n+1, F)$ of $(n+1) \times (n+1)$ trace zero matrices over F, where "#" is a commutative product in $sl(n+1, F)$ defined by

$$x \# y = xy + yx - \frac{2}{n+1} (Tr\ xy) I$$

Here, xy is the matrix product and Tr x is the trace of x. (15)

Following work of Benkart [2] for Lie-admissible algebras, in this section, we investigate the structure of Malcev-admissible algebras by weakening the flexible law to the condition

$$(x*x)*x = x*(x*x), \quad x \in A. \tag{16}$$

An algebra satisfying (16) is called a *third power (associative) algebra.* Clearly, any flexible and so commutative algebras are third power-associative. Related to this is fourth power-associativity

$$(x*x)*(x*x) = x*(x*(x*x)). \tag{17}$$

An algebra A is termed *power-associative* if the subalgebra of A generated by each element of A is associative. It is a known fact that if F is of characteristic 0 then (16) and (17) are sufficient to ensure that an algebra is power-associative. It has also been proved in [7] that the commutative product $x\#y$ defined by (15) is not fourth power-associative.

Let A be an algebra over a field F where A^- has any prescribed anticommutative product $[\ ,\ ]$. Let τ be a linear functional on A. We define a product $x*y$ on A by

$$x*y=\frac{1}{2}[x,y]+\tau(x)y+\tau(y)x \tag{18}$$

for x, $y\in A$. It is easy to see that the algebra A with product $x*y$ is power-associative and, furthermore $x^m=2^{m-1}\tau(x)^{m-1}x$, $x\in A$. Thus the subalgebra of A generated by x is at most one-dimensional.

We show that any third power Malcev-admissible algebra A with A^- isomorphic to C_0^- is described by the product (18).

THEOREM 4. *Let Abe a third power Malcev-admissible algebra over a field F of characteristic$\neq 2$ such that A^- is isomorphic to C_0^- Then the multiplication $x*y$ in A is given by* (18) *for some linear functional τ on A.*

Proof. We first note that third power-associativity (16) can be linearized to the identity

$$[x,\ y\circ z]=[x\circ y,\ z]+[x\circ z,\ y] \tag{19}$$

for all x, y, $z\in A$, where $x\circ y=\frac{1}{2}(x*y+y*x)$. We show that $x\circ y=\tau(x)y+\tau(y)x$ for all $x,y\in A$ and some linear functional τ on A. Using the basis h, e_i, e_{-i} $(i=1,2,3)$ given by (11), let

$$e_i\circ e_j=c^0{}_{ij}e_0+\sum_{k=1}^{3}(c^k{}_{ij}e_k+c^{-k}{}_{ij}e_{-k}), \tag{20}$$
$$i,\ j=0,\ \pm1,\ \pm2,\ \pm3,$$

where $c^k{}_{ij}\in F$ are the structure constants for A^+ and we have set $e_0=h$.

For $i\neq 0$, substituting (20) into the equation $[e_0\circ e_i,\ e_0]=[e_0,\ e_0\circ e_i]+[e_i,\ e_0\circ e_0]$ and using (11), we have

$$\begin{aligned}-\alpha\sum c^k{}_{0i}e_k+\alpha\sum c^{-k}{}_{0i}e_{-k}=&\,\alpha\sum c^k{}_{0i}e_k-\alpha\sum c^{-k}{}_{0i}e_{-k}\\ &-\alpha c^0{}_{00}e_i+\sum c^k{}_{00}[e_ie_k]+\sum c^{-k}{}_{00}[e_i,e_{-k}],\end{aligned} \tag{21}$$

where all summations are taken over the repeated index $k=1,2,3$. Since only one term involving $e_0=h$ occurs in one of the last two summations, one gets

$$c^t{}_{00}=0 \text{ for all } t\neq 0. \tag{22}$$

Hence, this can be combined with (21) to yield

$$c^i{}_{0i}=\frac{1}{2}c^0{}_{00},\quad c^t{}_{0i}=0 \text{ for all } t\neq i,\ i\neq 0,$$

and this with (22) gives

$$h\circ e_i=c^0{}_{0i}h+\frac{1}{2}c^0{}_{00}e_i,\quad i\neq 0,\quad h\circ h=c^0{}_{00}h. \tag{23}$$

Assume now that $i\neq 0$ and $j\neq 0$. From $[e_0, e_i\circ e_j]=[e_0\circ e_i, e_j]+[e_0\circ e_j, e_i]$ and (23), it follows that

$$\alpha\sum c^k{}_{ij}e_k-\alpha\sum c^{-k}{}_{ij}e_{-k}=c^0{}_{0i}[e_0, e_j]+c^0{}_{0j}[e_0, e_i]$$

and hence $c^0{}_{0i}=c^j{}_{ij}$, $c^t{}_{ij}=0$ for all $t\neq 0$, $\neq i$, $\neq j$. Therefore, by (20), we have

$$e_i\circ e_j=c_{ij}e_0+c_{0j}e_i+c_{0i}e_j,\quad i\neq 0,\ j\neq 0, \tag{24}$$

where we have set $c^0{}_{ij}=c_{ij}$. Finally, substituting (24) into the equation $[e_i\circ e_j, e_i]=[e_i, e_i\circ e_j]+[e_j, e_i\circ e_i]$ gives $c_{ij}=0$ for all $i\neq 0$, $j\neq 0$. Thus (24) reduces to

$$e_i\circ e_j=e_{0j}e_i+c_{0i}e_j,\quad i\neq 0,\ j\neq 0. \tag{25}$$

We define a linear functional τ on A by the rule

$$\tau(h)=\frac{1}{2}c_{00},\quad \tau(e_i)=c_{0i},\quad i=\pm 1,\ \pm 2,\ \pm 3.$$

Then, by (23) and (25), we have

$$e_i\circ e_j=\tau(e_i)e_j+\tau(e_j)c_i$$

for all i, j and hence the bilinearity of the product "$\circ$" gives the desired multiplication "$*$" for A, completing the proof.

Since C_0^- is the unique non-Lie, simple Malcev algebra over an algebracially closed field, combining Theorem 4 with the result in [1] for Lie admissible algebras, we have

THEOREM 5. *Let A be a finite-dimensional third power, Malcev-admissible algebra with product "$*$" over an algebraically closed field F of characteristic* 0 *such that A^- is simple over F. Then there is a linear functional $\tau : A\to F$ and a scalar $\beta\in F$ such that*

$$x*y=\frac{1}{2}\ [x, y]+\tau(x)y+\tau(y)x+\beta x\# y \tag{26}$$

where $x\# y$ is given by (15) *if A^- is a Lie algebra of type $A_n (n\geq 2)$, and $\beta x\# y=0$ for all $x, y\in A$, otherwise. Furthermore, A under product "$*$" is power-associative if and only if $\beta x\# y=0$ for all $x, y\in A$.*

Conversely, any multiplication "$*$" on a finite-dimensional simple Malcev algebra defined by (26) *using a linear functional τ and a scalar β determines a third power Malcev-admissible algebra, which is power-associative if and only if $\beta x\# y=0$ for all x, $y\in A$.*

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