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EXTENSIONS OF CONVERGENCE SPACES

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§0. Introduction

Since the topological structure on a topological space is completely determined by the data of the convergence of filters on the space, the convergence structure has been introduced to generalize the topological structure ([3], [4]). It is now well known that the category **Cv** of convergence space is a cartesian closed topological category which contains the category **Top** of topological spaces as a bireflective subcategory (see [2]) and that **Cv** is really a nice generalization of **Top**.

In topology, the extension theory is one of the most important and oldest topics, for extending a space to a space with nice properties, one can get more information of the original space and vice versa. In [1], extensions of topological spaces are in a quite extensive range established.

In this paper, it is our aim to give some systematic approach to the extensions of convergence spaces in the way introduced in [1].

Since in a convergence space there is no concept like the filter trace in [1], one has some difficulty to retrieve an extension of a convergence space by filters on the space, unless it is pretopological. However, we construct two typical extensions, namely strict and simple ones of a space associated with a certain family of filters, and then for a given extension, we try to compare it with those.

For the general terminologies of convergence spaces not introduced in this paper, we refer to [3] and [4] except that limit spaces in [3] are called here convergence spaces.

§1. Extensions of convergence spaces

For a set X, let $P(X)$ and $F(X)$ denote the power set of X and the set of all filters on X, respectively.

1.1 DEFINITION. For a set X, a map $c: X \to P(F(X))$ is called a *convergence structure* on X if it satisfies the following:

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1) for any $x \in X$, $\dot{x} \in c(x)$, where $\dot{x}$ denotes the principal filter generated by $\{x\}$.

2) if $\mathcal{F} \in c(x)$ and $\mathcal{F} \subseteq \mathcal{G} \in F(X)$, then $\mathcal{G} \in c(x)$.

3) if $\mathcal{F}$, $\mathcal{G} \in c(x)$, then $\mathcal{F} \cap \mathcal{G} \in c(x)$.

If c is a convergence structure on X, then (X, c) is called a convergence space.

1.2 DEFINITION. Let (X, c) be a convergence space.

1) If $\mathcal{F} \in c(x)$, then x is said to be a limit of $\mathcal{F}$, or $\mathcal{F}$ is said to *converge* to x, and we write $\mathcal{F} \to x$.

2) If any filter on X has at most one limit, then the space (X, c) is said to be a *Hausdorff convergence space.*

3) For $A \subseteq X$, the set $\{x \in X \mid$ there is a filter $\mathcal{F}$ converging to x and $A \in \mathcal{F}\}$ is called the *closure* of A and denoted by $\bar{A}$.

1.3 REMARK. For a convergence space, it is well known that the closure operator defined in the above satisfies the Kuratowski's axioms except $\bar{\bar{A}} = \bar{A}$.

1.4 DEFINITION. Let (X, c) and (Y, c') be convergence spaces and $f: X \to Y$ a map. Then f is said to be *continuous* on (X, c) to (Y, c') if for any filter $\mathcal{F} \in c(x)$, $f(\mathcal{F}) \in c'(f(x))$.

It is then obvious that the class of all convergence spaces and continuous maps between them forms a category which will be denoted by **Cv**.

It is well known ([2]) that **Cv** is a properly fibred cartesian closed topological category.

1.5 DEFINITION. Let (X, c) be a subspace of a convergence space (T, c'). Then the space (T, c') is said to be an *extension* of (X, c) if $\bar{X} = T$.

1.6 REMARK. If T is an extension of X, then for any $t \in T$, there is a filter on X converging to t.

Contrary to the extensions of a topological space, one can easily expect that an extension of an extension of a convergence space X need not be an extension of X, i.e., the transitivity of extensions does not hold.

We give here one example for the further development.

1.7 EXAMPLE. Let $X = \mathbf{R} \times \mathbf{R}$ and for any $(a, b) \in X$, let $\mathcal{U}((a, b))$ be the filter generated by $\{C_\varepsilon(a, b) \mid \varepsilon > 0\}$, where $C_\varepsilon(a, b) = \{(a, y) \mid |b - y| < \varepsilon\} \cup \{(x, b) \mid |x - a| < \varepsilon\}$.

We define a filter $\mathcal{F}$ on X converge to (a, b) iff $\mathcal{F}$ contains $\mathcal{U}((a, b))$. Then it is straightforward that X is a convergence space, more precisely a pretopological space but not a topological space. Let $S=[0,1]\times[0,1]$, $T=S-\{(0,0),\ (0,1),\ (1,0),\ (1,1)\}$ and $X=]0,\ 1[\times]0,\ 1[$. Then $\bar{X}=T$ and $\bar{T}=S$ so that S is an extension of T and T is an extension of X. But S is not an extension of X.

In order to get the data of an extension T of a given topological space X, the trace filters on X of neighborhood filters of points of T are excellently employed in [1]. If an extension of a convergence space is not pretopological, then one can not have the concept of trace filters. However, as already mentioned in Remark 1.6, for an extension T of a convergence space X, one can associate a filter on X to each point of T. Among others, an ultrafilter on X converging to apoint of T is a possible candidate, and if T is a Hausdorff pretopological space, then the set of T may be recovered by the set of all trace filters of points of T.

Let X be a convergence space and T be a superset of X, i.e., $X\subseteq T$. Consider a family $\{\mathfrak{F}_t \mid t\in T\}$ of filters on X such that for $x\in X$, $\mathfrak{F}_x=\dot{x}$ and for $t\in T-X$, $\mathfrak{F}_t$ is not convergent.

For any set $A\subseteq X$, let $\hat{A}=\{t\in T \mid A\in\mathfrak{U}_t\}$, then the following is immediate:

1.8 PROPORITION. *Under the same notations as above, one has the following*:

1) $\hat{A}=\phi$ *iff* $A=\phi$.

2) $\hat{A}\cap X=A$ *and* $\hat{X}=T$.

3) $\hat{A}\cap\hat{B}=\widehat{A\cap B}$; $\hat{A}\cup\hat{B}\subseteq\widehat{A\cup B}$.

4) *if each* $\mathfrak{F}_t$ $(t\in T)$ *is an ultrafilter, then* $\hat{A}\cup\hat{B}=\widehat{A\cup B}$.

In particular, for any filter $\mathfrak{F}$ on X, $\hat{\mathfrak{F}}=\{\hat{F} \mid F\in\mathfrak{F}\}$ forms a filter base on T. Using the above proposition, one can define a convergence structure $\hat{c}$ on T as follows:

For a filter Φ on T, $\Phi\xrightarrow{\hat{c}}x\in X$ iff there is a filter $\mathfrak{F}$ converging to x in X with $\hat{\mathfrak{F}}\subseteq\Phi$;

$$\Phi\xrightarrow{\hat{c}}t\in T-X \text{ iff } \hat{\mathfrak{F}}_t\subseteq\Phi.$$

1.9 PROPOSITION. *The convergence space* $(T,\hat{c})$ *is an extension of* X. *Furthermore, if* X *is a pretopological space, then* $(T,\hat{c})$ *is a pretopological extension of* X.

Proof. It is straightforward that $\hat{c}$ is indeed a convergence structure on T, and we omit the proof.

For any filter $\mathfrak{F}$ on X, the trace filter $\hat{\mathfrak{F}}_X$ of $\hat{\mathfrak{F}}$ on X is $\mathfrak{F}$ itself, so that a filter $\mathfrak{F}$ on X converges to x in X iff the filter $[\mathfrak{F}]$ on T generated by $\mathfrak{F}$

converges to x in T. Hence X is a subspace of T. For any $t \in T-X$, the filter base $\mathfrak{F}_t$ on T converges to t and hence T is an extension of X. If X is a pretopological space, and for each $x \in X$, $\mathcal{U}_x$ is the neighborhood filter of x, then, by the definition of $\hat{c}$, $\mathfrak{F}_t$ $(t \in T-X)$ and $\hat{\mathcal{U}}_x$ $(x \in X)$ are clearly neighborhood filters in T. Thus T is also a pretopological space.

1.10 DEFINITION. The convergence space T defined in the above thorem is called the *strict extension* of X associated with the family $\{\mathfrak{F}_t | t \in T\}$ of filters on X.

1.11 COROLLARY. *Let T be a strict extension of X associated with the family $\{\mathfrak{F}_t | t \in T\}$ of filters. For any $A \subseteq X$, the closure $\overline{A}^T$ of A in T contains $\overline{A}$.*

Proof. Since $B \subseteq \hat{B}$ for any $B \subseteq \hat{B}$, the filter base $\mathfrak{F}_t$ in T converges to t in T. If $t \in \overline{A}$, then the filter $[\mathfrak{F}_t]$ on T generated by $\mathfrak{F}_t$ converges to t in T and $A \in [\mathfrak{F}_t]$; therefore $t \in \overline{A}^T$.

1.12 PROPOSITION. *Suppose X is a hausdorff convergence space and T is a strict extension of X associated with a family $\{\mathfrak{F}_t | t \in T\}$ of filters. If every $\mathfrak{F}_t$ is an ultrafilter, then T is a Hausdorff extension of X.*

Proof. Suppose each $\mathfrak{F}_t$ is an ultrafilter and a filter Φ on T converges to two different points t and t'. If t, $t' \in T-X$, then $\mathfrak{F}_t$, $\mathfrak{F}_{t'} \subseteq \Phi$. Since $\mathfrak{F}_t$ and $\mathfrak{F}'_t$ are two different ultrafilters, there is $A \in \mathfrak{F}_t$ with $X-A \in \mathfrak{F}_t$ and hence $\hat{A}$, $\widehat{X-A} \in \Phi$. Since $\hat{A} \cap (\widehat{X-A}) = \Phi$, one has a contradiction. If t, $t' \in X$, there is a filter $\mathfrak{F}$ converging to t on X and a filter $\mathcal{G}$ converging to t' on X such that $\hat{\mathfrak{F}}$, $\hat{\mathcal{G}} \subseteq \Phi$. Thus there is a filter $\mathcal{H}$ containing $\mathfrak{F}$ and $\mathcal{G}$ and hence one has again a contradiction to the fact that X is Hausdorff. Finally, if $t \in X$ and $t' \in T-X$, then there is a filter $\mathfrak{F}$ converging to t on X with $\mathfrak{F}_t$, $\mathfrak{F}_{t'} \subseteq \Phi$. Again there is a filter $\mathcal{H}$ containing $\mathfrak{F}$ and $\mathfrak{F}_{t'}$ so that $\mathfrak{F} \subseteq \mathfrak{F}_{t'} = \mathcal{H}$. Since $\mathfrak{F}_{t'}$ is not convergent, one has a contradiction.

This completes the proof.

Using the above proposition, one has the interesting characterization of compact Hausdorff convergence space which is quite different from that of compact Hausdorff topological spaces.

1.13 DEFINITION. A Hausdorff convergence space is said to be *H-closed* if it is closed in any Hausdorff convergence space containing X as a subspace.

1.14 THEOREM. *A Hausdorff convergnce space is compact iff it is H-closed.*

Proof. Let X be a Hausdorff compact convergence space. Suppose X is a subspace of a Hausdorff convergence space Y and x belongs to the closure $\bar{X}$ of X in Y. Then there is an ultrafilter $\mathcal{U}$ on Y such that $\mathcal{U} \longrightarrow x$ and $X \in \mathcal{U}$. Since $X \in \mathcal{U}$, the trace filter $\mathcal{U}_x$ of $\mathcal{U}$ on X is an ultrafilter on X. Since X is compact, $\mathcal{U}_x$ converges to some point $a \in X$ and hence $\mathcal{U}$ converges to a. Since Y is Hausdorff, $a = x$, i. e., $x \in X$. Thus $\bar{X} = X$, i. e., X is closed in Y. Conversely, suppose X is H-closed and not compact. Then there is a non-convergent ultrafilter, say $\mathcal{U}$ on X. Let $T = X \cup \{w\}$, where $w \notin X$, and consider $\{\mathcal{F}_t \mid t \in T\}$ such that $\mathcal{F}_x = \dot{x}$ $(x \in X)$ and $\mathcal{F}_w = \mathcal{U}$. Then the strict extension T of X associated wih $\mathcal{F}_t \mid t \in T\}$ is Hausdorff by the above proposition. Since X is H-closed, $T = \bar{X} = X$ and hence $w \in X$, which is a contradiction. This completes the proof.

1.15 REMARK. We note that a H-closed topological space need not be compact.

Now let us return to any extension T of a convergence space X. Firstly we should remark that if instead of $\dot{x}$ $(x \in X)$ we choose the neighborhood filters for a pretopological space X and endow the convergence structure as that in Theorem 1.9, then T need not be an extension of X.

1.16 EXAMPLE. Let X and T be the spaces introduced in Example 1.7. Then for each $t \in T - X$, choose $\mathcal{F}_t$ to be the trace filter of $\mathcal{N}(t)$ on X and let $\mathcal{F}_x = \mathcal{N}(x)$ for $x \in X$. Take a point $t = (a, 0) \in T - X$, then $\mathcal{F}_t$ is generated by $\{\{(a, y) \mid 0 < y < \varepsilon\} \mid \varepsilon > 0\}$ and $\mathcal{N}(x)$ is generated by the "cross" sets. Hence $\mathcal{F}_t = \dot{t}$. Hence t does not belong to the closure of X in the space T_1 constructed as above. Thus T_1 is not an extension of X.

For a Hausdorff extension T of a space X, let us choose any family $\{\dot{x} \mid x \in X\} \cup \{\mathcal{F}_t \mid \mathcal{F}_t$ is an ultrafilter on X converging to t in $T; t \in T - X\}$ unless T is a pretopological space. If T is a Hausdorff pretopological space, then we chooes the family $\{\dot{x} \mid x \in X\} \cup \{\mathcal{F}_t \mid \mathcal{F}_t$ is the trace filter of the neighborhood filter of t in T; $t \in T - X\}$. Then we shall compare T and the strict extension sX of X associated with the above family of filters. We do not know yet the identity map $sX \longrightarrow T$ is continuous as that of the strict extension of a topological space is [1].

1.17 DEFINITION. A convergence space X is said to be *regular* if it satisfies the following: a filter $\mathcal{F}$ converges to x iff $\bar{\mathcal{F}} = \{\bar{F} \mid F \in \mathcal{F}\}$ converges to x.

1.18 THEOREM. *If T is a regular extension of X, then the identity map $sX \longrightarrow T$ is continuous.*

Proof. Let Φ be a filter converging to t in sX. If $t \in X$, then there is a filter $\mathfrak{F}$ on X converging to t with $\hat{\mathfrak{F}} \subset \Phi$. Since $\mathfrak{F}$ converges to t in X, $\mathfrak{F}$ also converges to t in T. Since T is regular, $\{\bar{F}^T \mid F \in \mathfrak{F}\}$ also converges to t. By Corollary 1.11, for any $F \in \mathfrak{F}$, $\hat{F} \subseteq \bar{F}^T$ and since $\hat{F} \subset \Phi$, $\{\bar{F}^T \mid F \in \mathfrak{F}\}$ is contained in Φ and hence Φ converges to t in T. If $t \in T-X$, then $\mathfrak{F}_t$ converges to t in T and $\mathfrak{F}_t \subseteq \Phi$. Again using Corollary 1.11, one has $\{\bar{F}^T \mid F \in \mathfrak{F}_t\} \subseteq \Phi$ and hence Φ converges to t in T.

This completes the proof.

§2. Simple extensions

For a pretopological space (X, c) and a superset T of X, let us consider a family $\{\mathfrak{F}_t \mid t \in T\}$ of filters on X such that for $t \in X$, $\mathfrak{F}_t$ is the neighborhood filter $\mathcal{U}(t)$ of t and for $t \in T-X$, $\mathfrak{F}_t$ is a nonconvergent filter. We define a convergence structure $\bar{c}$ as follows; for a filter Φ on T and $t \in T$, $\Phi \xrightarrow{\bar{c}} t$ iff for any $F \in \mathfrak{F}_t$, $F \cup \{t\} \in \Phi$.

2.1 THEOREM. *The space* $(T, \bar{c})$ *is a pretopological extension of* (X, c).

Proof. It is clear that $\bar{c}$ is a pretopological convergence structure on T and we omit the proof. Take any filter $\mathfrak{F}$ on X. Then $\mathfrak{F} \xrightarrow{c} x$ iff $\mathcal{U}(x) \subseteq \mathfrak{F}$. Since $\dot{x} \longrightarrow x$ in X, x belongs to each member of $\mathcal{U}(x)$ and hence for any $F \in \mathcal{U}(x)$, $F \cup \{x\} = F$. Thus $\mathfrak{F} \xrightarrow{c} x$ iff the filter $[\mathfrak{F}]$ on T generated by $\mathfrak{F}$ converges to x in T. Hence X is a subspace of T. Moreover, for any $t \in T-X$, the filter $[\mathfrak{F}_t]$ on T generated by $\mathfrak{F}_t$ converges to t, i.e., $t \in \bar{X}$.

This completes the proof.

2.2 REMARK. If X is Hausdorff and for $t \in T-X$, $\mathfrak{F}_t$ is an ultrafilter on X, then the extension $(T, \bar{c})$ of X is again Hausdorff.

2.3 DEFINITION. The convergence space T defined in the above theorem is called the *simple extension* of X associated with the family $\{\mathfrak{F}_t \mid t \in T\}$ of filters on X.

For any Hausdorff pretopological extension T of X, let $\mathfrak{F}_t$ be the trace filter of the neighborhood filter $\mathcal{U}(t)$ of t on X and let pX be the simple extension associated with $\{\mathfrak{F}_t \mid t \in T\}$.

The following theorem amounts to saying that the convergence structure on a Hausdorff pretopological extension is weaker than that on the simple extension.

2.4 THEOREM. *The identity map $pX \longrightarrow T$ is continuous.*

Proof. Suppose that a filter Φ converges to t in pX. Take any $N \in \mathscr{U}(t)$, then $U = N \cap X \in \mathscr{F}_t$. Since $\Phi \xrightarrow{\bar{c}} t$, $U \cup \{t\}$ belongs to Φ, and hence N also belongs to Φ. Thus $\mathscr{U}(t) \subset \Phi$, so Φ converges to t in T.

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