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ON ORDERED HYPERSPACES

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0. Introduction

In [1], Choe and the first author introduced the concept of an ordered hyperspace and showed that a regularly ordered C-space with a semicontinuous order can be embedded into a bounded topological semilattice.

In this paper, we will investigate relationships between an ordered topological space and its ordered hyperspace, and between continuous increasing functions on ordered topological spaces and continuous increasing set-valued functions on ordered hyperspaces, which reduce the corresponding results of topological spaces whenever given the order is discrete([3], [5]). Let $(X, \leq)$ be a partially ordered set and A a subset of X; then we write

$$d(A)=\{y\in X : y\leq x \text{ for some } x\in A\},$$
$$i(A)=\{y\in X : x\leq y \text{ for some } x\in A\}.$$

In particular, if A is a singleton, say $\{x\}$, then we write $d(x)$ (resp. $i(x)$) instead of $d(\{x\})$ (resp. $i(\{x\})$). A subset A of X is said to be *decreasing* (resp. *increasing*) if $A=d(A)$ (resp. $A=i(A)$). The order is called *discrete* if $x\leq y$ only when $x=y$. By an *ordered topological space* we mean a set X endowed with both a topology $\mathcal{T}$ and a partial order $\leq$.

For an ordered topological space $(X, \mathcal{T}, \leq)$, let

$$\mathcal{U}=\{U\in\mathcal{T} : U=i(U)\}, \qquad \mathcal{L}=\{U\in\mathcal{T} : U=d(U)\},$$

then $\mathcal{U}$ and $\mathcal{L}$ are evidently topologies for X, which are called the *upper*, *lower* topologies respectively ([7]). We say that an ordered topological space X is *convex* if X has a subbase consisting of the sets in $\mathcal{U}$ and $\mathcal{L}$, or equivalently, if every open sets in X can be written as the intersection of an open decreasing set and an open increasing set ([6]).

The order is *lower* (*upper*) *semicontinuous* provided, $x\not\leq y$ ($y\not\leq x$) in X, there is an open set U with $x\in U$, such that if $u\in U$ then $u\not\leq y$ ($y\not\leq u$). The order is *semicontinuous* if it is both upper and lower semicontinuous. It is *continuous* provided, whenever $x\not\leq y$ in X, there are open sets U and V, $x\in U$ and $y\in V$, such that if $u\in U$ and $v\in V$, then $u\not\leq v$.

For general topological background and terminology, we refer to [3] and

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for ordered topological spaces to [6], [8].

1. Relationships between an ordered topological space and its ordered hyperspace

The concepts of an ordered hyperspace was introduced by [1] as follows. Let $(X, \mathscr{T}, \leq)$ be an ordered topological space. Let $D(X)$ denote the set of all closed, decreasing subsets of X, For a subset A of X, we let

$$\Gamma(A)=\{F\in D(X) : F\subseteq A\}.$$

Consequently,

$$D(X)-\Gamma(X-A)=\{F\in D(X) : F\cap A\neq\emptyset\}.$$

Let $\mathscr{T}^*$ be the smallest topology on $D(X)$ generated by the family $\{\Gamma(U) : U$ is an open decreasing subset of $X\}\cup\{D(X)-\Gamma(X-U) : U$ is an open increasing subset of $X\}$.

We shall use the following notational convention. For subsets $A_0, A_1, \ldots, A_n$ of X, we let

$$B(A_0;A_1, \ldots, A_n)=\Gamma(A_0)\bigcap_{i=1}^{n}(D(X)-\Gamma(X-A_i))$$
$$=\{F\in D(X) : F\subseteq A_0 \text{ and } F\cap A_i\neq\emptyset \text{ for all } i=1, 2, \ldots, n\}.$$

Using this notation, the sets $B(U_0; U_1, \ldots, U_n)$, where U_0 is an open decreasing and $U_1, \ldots, U_n$ are open increasing in X, form an open base of $D(X)$. Hence $(D(X), \mathscr{T}^*, \subseteq)$ is an ordered topological space, where $\subseteq$ is the set inclusion relation.

REMARKS 1. (1) Dually, we can define an ordered hyperspace $(I(X), \mathscr{T}_*, \subseteq)$ where $I(X)$ denotes the set of all closed increasing subsets of X and $\mathscr{T}_*$ is the smallest topology on $I(X)$ generated by the family $\{\Gamma(U) : U$ is an open increasing subset of $X\}\cup\{I(X)-\Gamma(X-U) : U$ is an open decreasing subset of $X\}$. [1]

(2) If A is a closed decreasing (resp. closed increasing) subset of X, then $\Gamma(A)$ (resp. $D(X)-\Gamma(X-A)$) is closed decreasing (resp. closed increasing) in $D(X)$.

(3) Only nonempty basic open, increasing sets in $D(X)$ are of the form $B(X; U_1, \ldots, U_n)$, where $U_1, \ldots, U_n$ are open increasing sets in X. Dually, only nonempty basic open, decreasing sets in $D(X)$ are of the form $\Gamma(U)$, where U is open decreasing in X. Hence $D(X)$ is a convex ordered topological space.

(4) If given the order on X is discrete, then $D(X)$, $I(X)$ are the hyperspace of X([3], [5]) and $D(X)=I(X)$.

PROPOSITION 2. *Let X be an ordered topological space with a lower semicontinuous order. Then the family $\{d(F) : F$ is a finite subset of $X\}$ is a* *dense subset of* $D(X)$.

Proof. Let $B(U_0;\ U_1, \dots, U_n)$ be a non-empty basic open set in $D(X)$. Then there exists an $A \in D(X)$ such that $A \subseteq U_0$ and $A \cap U_i \neq \varnothing$ for each $i = 1, 2, \dots, n$. Choosing $x_0 \in U_0$ and $x_i \in A \cap U_i$ for each i and letting $K = \{x_0, x_1, \dots, x_n\}$, we have $d(K) \in B(U_0;\ U_i, \dots, U_n)$. Thus $\{d(F) : F$ is a finite subset of $X\}$ is dense in $D(X)$.

LEMMA 3. *Let* X *be an ordered topological space with an upper semicontinuous order and* A *a subset of* X. *Then the family* $\{G \in D(X) : A \subseteq G\}$ *is closed in* $D(X)$.

Proof. It is easy to see that

$$D(X) - \{G \in D(X) : A \subseteq G\} = \bigcup_{x \in A} \{G \in D(X) : G \subseteq X - i(x)\}.$$

Since the order is upper semicontinuous, $X - i(x)$ is open, decreasing. Thus $\{G \in D(X) : G \subseteq X - i(x)\}$ is open and hence $\{G \in D(X) : A \subseteq G\}$ is closed.

THEOREM 4. *Let* X *be an ordered topological space with a semicontinuous order. Then* $D(X)$ *is an ordered topological space with the semicontinuous order.*

Proof. Let $K \in D(X)$. Then $d(K) = \Gamma(K)$ and $i(K) = \{G \in D(X) : K \subseteq G\}$. It follows from Remark 1 and Lemma 3 that $d(K)$ and $i(K)$ are closed. Hence the order is semicontinuous.

DEFINITION 5. (McCartan [4]) An ordered topological space X is said to be *lower* (*upper*) *regularly ordered* if and only if for each decreasing (increasing) closed set $G \subseteq X$ and each element $x \notin G$, there exist disjoint open neighborhoods U of x and V of G such that U is increasing (decreasing) and V is decreasing (increasing) in X. It is said to be *regularly ordered* if it is both lower and upper regularly ordered.

THEOREM 6. *Let* X *be an ordered topological space with a lower semicontinuous order. Then* X *is lower regularly ordered if and only if* $D(X)$ *is an ordered topological space with the continuous order.*

Proof. ($\Longrightarrow$) Let $G, K \in D(X)$ with $G \not\subseteq K$. Then there exists an $x \in G$ such that $x \notin K$. Since X is lower regularly ordered, there are disjoint open increasing neighborhood U of x and open decreasing neighborhood V of K in X. It is easy to show that $\Gamma(V)$ is an open decreasing neighborhood of K and $B(X;\ U)$ is an open increasing neighborhood of G in $D(X)$ such that $\Gamma(V) \cap B(X;\ U) = \varnothing$. Thus the order on $D(X)$ is continuous.

($\Longleftarrow$) Let $x \in X$ and let G be a decreasing closed set with $x \notin G$. Then $G \cup d(x) \not\subseteq G$. Let B_1 and B_2 be disjoint open decreasing neighborhood of G and open increasing neighborhood of $G \cup d(x)$ in $D(X)$ respectively. Then there are basic open sets $B(U_0;\ U_1, \ldots, U_m)$ and $B(V_0;\ V_1, \ldots, V_n)$ such that $G \in B(U_0;\ U_1, \ldots, U_m) \subseteq B_1$ and $G \cup d(x) \in B(V_0;\ V_1, \ldots, V_n) \subseteq B_2$. It follows that there exists an $j\,(1 \leq j \leq n)$ such that $x \in V_j$ and $U_0 \cap V_0 \cap V_j = \emptyset$. Since $G \subseteq U_0 \cap V_0$, X is lower regularly ordered.

DEFINITION 7. (Priestley [7]) (1) An ordered topological space X is called an *C_i-space* (resp. *C_d-space*) if, whenever a subset G of X is closed $i(G)$ (resp. $d(G)$) is closed. If X is both, then X is called a *C-space.*

(2) An ordered topological space X is called an *I_i-space* (resp. *I_d-space*) if, whenever a subset U of X is open, so is $i(U)$ (resp. $d(U)$). If X is both an I_i and I_d-space, it is called an *I-space.*

PROPOSITION 8. *Let X be an C_i-space. If X is lower regularly ordered. then $\{(K,\ L) \in D(X) \times D(X) : K \subseteq L\}$ and $\{(x,\ K) \in X \times D(X) : x \in K\}$ are closed in $D(X) \times D(X)$, respectively in $X \times D(X)$.*

Proof. Let $(K,\ L) \in D(X) \times D(X)$ with $K \not\subseteq L$; then there is an $x \in K$ such that $x \notin L$. Hence there exist an open increasing neighborhood U of x and an open decreasing neighborhood V of L such that $U \cap V = \emptyset$. It follows that $i(\bar{U}) \cap L = \emptyset$ and $L \subseteq X - i(\bar{U})$. Thus

$$\{(K,\ L) : K \not\subseteq L\} = \bigcup_U (\{K : U \cap K \neq \emptyset\} \times \{L : L \subseteq X - i(\bar{U})\}).$$

Since X is an C_i-space, $X - i(\bar{U})$ is an open decreasing set. Hence, $\{(K, L) : K \not\subseteq L\}$ is open. It follows that $\{(K,\ L) \in D(X) \times D(X) : K \subseteq L\}$ is closed in $D(X) \times D(X)$. Similarly, $\{(x,\ K) \in X \times D(X) : x \in K\}$ is closed in $X \times D(X)$.

THEOREM 9. *Let X be an I_i-space with a lower semicontinuous order. If $\{(x,\ X) \in X \times D(X) : x \in K\}$ is closed in $X \times D(X)$, then X is lower regularly ordered.*

Proof. Let K be a closed decreasing set of X and let $x \notin K$. Let $G = X \times D(X) - \{(y, F) \in X \times D(X) : y \in F\}$. Then G is an open neighborhood of (x, X). Hence there exists an open set U of X, an open set $B(V_0;\ V_1, \ldots, V_n)$ of $D(X)$ such that $(x,\ K) \in U \times B(V_0;\ V_1, \ldots, V_n) \subseteq G$. It follows that $U \cap V_0 = \emptyset$. Hence we have $i(U) \cap V_0 = \emptyset$. Hence we have $i(U) \cap V_0 = \emptyset$. Since $x \in i(U)$ and $K \subseteq V_0$, X is clearly lower regularly ordered.

DEFINITION 10. (Nachbin [6]) An ordered topological space X is said to be *normally ordered* if for every two disjoint closed subsets F_0 and F_1 of X such that F_0 is decreasing and F_1 is increasing, then there exist two disjoint open sets U_0 and U_1 such that U_0 contains F_0 and is decreasing, and U_1 contains F_1 and is increasing.

THEOREM 11. *Let X be an ordered topological space with semicontinuous order. Then X is normally ordered if and only if the set $\{(K, L)\in D(X)\times I(X) : K\cap L=\varnothing\}$ is open in $D(X)\times I(X)$.*

Proof. ($\Longrightarrow$) Assume that X is normally ordered. Then we have

$$\{(K,L) : K\cap L=\varnothing\}=\bigcup_{U\cap V=\phi}\{(K, L) : K\subseteq U \text{ and } L\subseteq V\}$$
$$=\bigcup_{U\cap V=\phi}(\Gamma(U)\times\Gamma(V)),$$

where U is open decreasing and V is open increasing in X. Since $\Gamma(U)$ and $\Gamma(V)$ are open in $D(X)$ and $I(X)$ respectively, $\{(K, L)\in D(X)\times I(X) : K\cap L=\varnothing\}$ is open in $D(X)\times I(X)$.

($\Longleftarrow$) Let F_0 ad F_1 be two disjoint closed subsets of X such that F_0 is decreasing and F_1 is increasing. Then $(F_0, F_1)\in\{(K, L)\in D(X)\times I(X) : K\cap L=\varnothing\}$, and hence there exist basic open sets $B_0(U_0; U_1, \ldots, U_m)$ and $B_1(V_0; V_1, \ldots, V_n)$ in $D(X)$ and $I(X)$ respectively, such that $(F_0, F_1)\in B_0(U_0; U_1, \ldots, U_m)\times B_1(V_0; V_1, \ldots, V_n)\subseteq\{(K, L) : K\cap L=\varnothing\}$. It is easy to show that $U_0\cap V_0=\varnothing$. Since $F_0\subseteq U_0$ and $F_1\subseteq V_0$, X is normally ordered.

REMARK 12. In the previous results, if given the order on X is discrete, then they reduce the corresponding ones of topological spaces ([3], [5]).

DEFINITION 13. (Priestly [7]) Let $(X, \mathscr{T}, \leq)$ be an ordesed topological space and Y a subset of X. Then an ordered topological space $(Y, \mathscr{T}|Y, \leq_Y)$ is called an *order subspace* of $(X, \mathscr{T}, \leq)$ if $\mathscr{U}|Y$, $\mathscr{L}|Y$ coincide with the upper, lower topologies of $(Y, \mathscr{T}|Y, \leq_Y)$, where $\mathscr{T}|Y$ is the induced topology, $\leq_Y$ is the induced order, and $\mathscr{U}$, $\mathscr{L}$ are upper, lower topologies of X, respectively.

Let $(2^X, \tau)$ be an hyperspace of a topological space $(X, \mathscr{T})$. Then $(2^X, \tau, \subseteq)$ is an ordered topological space, where $\subseteq$ is the set inclusion order. In the following, we will show that $(D(X), \mathscr{T}^*, \subseteq)$ is an order subspace of $(2^X, \tau, \subseteq)$ under some conditions of $(X, \mathscr{T}, \leq)$.

LEMMA 14. *Let $(X, \mathscr{T}, \leq)$ be a compact ordered space and an I_i-space. Then $\mathscr{T}^*=\tau|D(X)$.*

Proof. Let $\mathbf{B}$ be a nonempty basic open in $\tau|D(X)$. Then there exists a basic open set $B(U_0; U_1, \ldots, U_n)$ in τ such that $\mathbf{B}=B(U_0; U_1, \ldots, U_n)\cap D(X)$. Let $F\in\mathbf{B}$. Then $F\subseteq U_0$ and $F\cap U_i\neq\varnothing$ for each $i=1, 2, \ldots, n$. Since X is compact ordered, there exists an open decreasing neighborhood V_0 of F with $F\subseteq V_0\subseteq U_0$. It is easy to see that $F\in B(V_0; i(U_1), \ldots, i(U_n))\subseteq\mathbf{B}$ and $B(V_0; i(U_1), \ldots, i(U_n))$ is a basic open set in $\mathscr{T}^*$. Thus $\mathbf{B}\in\mathscr{T}^*$.

REMARKS 15. (1) Let $(X, \mathscr{T}, \leq)$ be a compact ordered space and an I_d-space.

Then $\mathscr{T}_* = \tau | I(X)$.

(2) The hypothesis of compactness in Lemma 14 is essential. For example, let N be the natural numbers with the discrete topology and the usual order. Then $\mathscr{T}^* \neq \mathscr{T} | D(N)$.

LEMMA 16. *Let $(X, \mathscr{T}, \leq)$ be a compact ordered space. Then $D(X) = \{F \in 2^X : F = d(F)\}$ is closed in $(2^X, \tau)$.*

Proof. Let f be the identity function on 2^X. Define a function $g : 2^X \to 2^X$ by $g(F) = d(F)$ for each $F \in 2^X$. Then g is continuous. Indeed, it is sufficient to show that a function $h : 2^X \to D(X)$ defined by $h(F) = d(F)$ is continuous. To see continuity of h, let $F \in 2^X$ and $B(U_0; U_1, \ldots, U_n)$ be a basic open neighborhood of $h(F)$ in $D(X)$. It follows that $B(U_0; U_1, \ldots, U_n)$ is an open neighborhood of F in 2^X and $h[B(U_0; U_1, \ldots, U_n)] \subseteq B(U_0; U_1, \ldots, U_n)$. Hence h is continuous. Since $\{F \in 2^X : F = d(F)\} = \{F \in 2^X : f(F) = g(F)\}$, the set $D(X)$ is clearly closed in $(2^X, \tau)$.

THEOREM 17. *Let $(X, \mathscr{T}, \leq)$ be a compact ordered space and an I_i-space. Then $(D(X), \mathscr{T}^*, \subseteq)$ is an order subspace of $(2^X, \tau, \subseteq)$.*

Proof. Immediate from Lemma 14 and 16, and Proposition 5 of [7].

COROLLARY 18. ([1]) *Let $(X, \mathscr{T}, \leq)$ be a compact zero-dimensional I-space with continuous order. Then $(D(X), \mathscr{T}^*, \cup)$ is a compact zero-dimensional semilattice, where $\cup$ is the set union.*

REMARKS 19. (1) Let $(X, \mathscr{T}, \leq)$ be a compact ordered space and an I_d-space. Then $(I(X), \mathscr{T}_*, \subseteq)$ is an order subspace of $(2^X, \tau, \subseteq)$.

(2) Let $(X, \mathscr{T}, \leq)$ be a compact zero-dimensional I-space with continuous order. Then $(D(X), \mathscr{T}^*, \subseteq)$ is a compact zero-dimensional ordered space.

2. Increasing continuous set-valued function

In this section, we will investigate relationships between continuous increasing functions on ordered topological spaces and continuous increasing set-valued functions on ordered hyperspaces.

THEOREM 20. *Let Y be an ordered topological space and let X be an ordered topological space with a lower semicontinuous order. Let $f : Y \to X$ be a continuous increasing function. Define $F : Y \to D(X)$ by $F(y) = d(f(y))$ for each $y \in Y$. Then F is a continuous increasing function.*

Proof. Clearly F is increasing. To show the continuity of F, let $B(U_0; U_1, \ldots, U_n)$ be a basic open neighborhood of $d(f(y))$. Then $d(f(y)) \subseteq U_0$ and

$d(f(y)) \cap U_i \neq \emptyset$ for each $i=1, 2, \ldots, n$. It follows that $f(y) \in U_i$ for all $i=0, 1, \ldots, n$. Let

$$U=\bigcap_{i=0}^{n} f^{-1}(U_i).$$

Then it is easy to see that U is an open neighborhood of y and $F(U) \subseteq B(U_0; U_1, \ldots, U_n)$. Thus F is continuous.

THEOREM 21. *Let X be an ordered topological space and let Y be an C-space. Let $f: X \to Y$ be acontinuous, closed and increasing function. Define $H: D(X) \to D(Y)$ by $H(F)=d(f(F))$ for each $F \in D(X)$. Then H is a continuous, increasing function.*

Proof. Let $F \in D(X)$ and let $B(U_0; U_1, \ldots, U_n)$ be a basic open neighborhood of $H(F)$ in $D(Y)$. It follows that $F \subseteq f^{-1}(U_0)$ and $F \cap f^{-1}(U_i) \neq \emptyset$ for each $i=1, 2, \ldots, n$. Since f is continuous, increasing, $f^{-1}(U_0)$ is open decreasing and $f^{-1}(U_i)$ is open increasing for all $i=1, 2, \ldots, n$. Hence $B(f^{-1}(U_0); f^{-1}(U_1), \ldots, f^{-1}(U_n))$ is an open neighborhood of F and $H(B(f^{-1}(U_0); f^{-1}(U_1), \ldots, f^{-1}(U_n)) \subseteq B(U_0; U_1, \ldots, U_n)$. Thus H is continuous; moreover, H is clearly increasing.

REMARK 22. If given the order in Theorem 20 and 21 is discrete, then they reduce the corresponding results of topological spaces ([3], [5]).

References

1. Choe, T. H. and Park, Y. S. *Embedding ordered topological spaces into topological semilattices*, Semigroup Forum **17** (1979) 188–199.
2. Ginsberg, J., *On the Stone-Čech compactification of the spaces of the closed sets*, Trans. Amer. Math. Soc. **215** (1976) 301–311.
3. Kuratowski, K., *Topology*, Academic Press, New York, 1966.
4. McCartan, S. D., *Separation axioms for topological ordered spaces*, Proc. Camb. Phil. Soc. **64** (1968) 965–973.
5. Michael, E., *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951) 152–183.
6. Nachbin, L., *Topology and order*, Van Nonstrand Mathematical Studies 4, Princeton, N. J., 1965.
7. Priestley, H. A., *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. (3) **24** (1972) 507–530.
8. Ward Jr., L. E., *Partially ordered topological spaces*, Proc. Amer. Math. Soc. **5** (1954) 144–166.

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