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# CONTACT-THREE-CR SUBMANIFOLDS

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## 1. Introduction

The contact CR submanifolds of a Sasakian manifold have been defined and studied by K. Yano and M. Kon [11] and are now being studied by many authors.

The main purpose of the present paper is to define what we call contact-three-CR submanifolds of a manifold with Sasakian-three-structure and to study their properties.

## 2. Submanifolds of a manifold with Sasakian-three-structure

In a Riemannian manifold $(\bar{M}, g)$ of dimension $m$ with metric tensor $g$, let there be given a Killing vector $\xi$ of unit length satisfying the condition

$$\bar{\nabla}_j\bar{\nabla}_i\xi^h=\xi_i\delta_j^h-\xi^h g_{ji}$$

$\xi^h$ being components of $\xi$ and $g_{ji}$ components of $g$, where $\xi_i=\xi^h g_{hi}$ and $\bar{\nabla}_j$ denote the operater of covariant differentiation with respect to the Riemannian connection of $(\bar{M}, g)$. Then $\xi$ is called a Sasakian structure in $(\bar{M}, g)$ (see [9]).

We assume that $(\bar{M}, g)$ admits three Sasakian structures $\xi$, $\eta$ and $\zeta$ which are mutually orthogonal and satisfy the conditions

$$[\eta, \zeta]=2\xi, \quad [\zeta, \xi]=2\eta, \quad [\xi, \eta]=2\zeta.$$

Then the set $\{\xi, \eta, \zeta\}$ is called a Sasakian-three-structure in $(\bar{M}, g)$. In such a case, $\bar{M}$ is necessarily of dimension $m=4n+3$ $(n\geqq 0)$. Moreover, the distribution $\mathcal{D}$ spanned by $\xi, \eta$ and $\zeta$ is integrable and every integral manifold of $\mathcal{D}$ is totally geodesic and of constant curvature 1 (see [5], [7]).

We denote by $\alpha, \beta$ and $\gamma$ the 1-forms associated with $\xi, \eta$ and $\zeta$ respectively. If we put

$$\phi=\bar{\nabla}\xi, \quad \psi\bar{\nabla}\eta, \quad \theta=\bar{\nabla}\zeta$$

then we can easily verify that

$$\begin{array}{lll} \phi\xi=0, & \psi\eta=0, & \theta\zeta=0, \\ \alpha\circ\phi=0, & \beta\circ\psi=0, & \gamma\circ\theta=0, \\ \theta\eta=-\psi\zeta=-\xi, & \phi\zeta=-\theta\xi=-\eta, & \psi\xi=-\phi\eta=-\zeta, \end{array}$$

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$$\phi^2=-I+\alpha\otimes\xi,\quad \psi^2=-I+\beta\otimes\eta,\quad \theta^2=-I+\gamma\otimes\zeta,$$
$$\psi\theta=\phi+\gamma\otimes\eta,\quad \theta\phi=\psi+\alpha\otimes\zeta,\quad \phi\psi=\theta+\beta\otimes\xi,$$
$$\theta\psi=-\phi+\beta\otimes\zeta,\quad \phi\theta=-\psi+\gamma\otimes\xi,\quad \psi\phi=-\theta+\alpha\otimes\eta, \tag{2.1}$$
$$\beta\circ\phi=-\alpha\circ\psi=-\gamma,\quad \gamma\circ\psi=-\beta\circ\theta=-\alpha,\quad \alpha\circ\theta=-\gamma\circ\phi=-\beta,$$

and

$$(\bar{\nabla}_X\phi)Y=-g(X,Y)\xi+\alpha(Y)X,$$
$$\bar{\nabla}(_X\psi)Y=-g(X,Y)\eta+\beta(Y)X, \tag{2.2}$$
$$(\bar{\nabla}_X\theta)Y=-g(X,Y)\zeta+\gamma(Y)X$$

for any vector fields $X$ and $Y$ on $\bar{M}$, where $I$ denotes the identity tensor field of type (1.1) with components $\delta_i^h$.

Let $M$ be a submanifold isometrically immersed in $\bar{M}$. Throughout this paper, we assume that the submanifold $M$ of $\bar{M}$ is tangent to the structure vectors $\xi,\eta$ and $\zeta$. For any vector field $X$ tangent to $M$. we put

$$\phi X=P_1X+F_1X,\quad \psi X=P_2X+F_2X,\quad \theta X=P_3X+F_3x, \tag{2.3}$$

where $P_rX(r=1,2,3)$ are the tangential parts and $F_rX(r=1,2,3)$ the normal parts of $\phi X$, $\psi X$ and $\theta X$ respectively. Then $P_r$ is an endomorphism on the tangent bundle $T(M)$ and $F_r$ is a normal bundle valued 1-form on $T(M)$. Similarly, for any vector field $V$ normal to $M$, we put

$$\phi V=t_1V+f_1V,\quad \psi V=t_2V+f_2V,\quad \theta V=t_3V+f_3V, \tag{2.4}$$

where $t_rV(r=1,2,3)$ are the tangential parts and $f_rV(r=1,2,3)$ the normal parts of $\phi V$, $\psi V$ and $\theta V$ respectively. For any vector field $Y$ tangent to $M$, we have from (2.3)

$$g(P_rX,Y)=-g(P_rY,X),\quad r=1,2,3 \tag{2.5}$$

because $\bar{\nabla}\alpha$, $\bar{\nabla}\beta$ and $\bar{\nabla}\gamma$ are skew-symmetric. Similarly for any vector field $U$ normal to $M$, we have from (2.4)

$$g(f_rV,U)=-g(f_rU,V),\quad r=1,2,3 \tag{2.6}$$

because of the same reason. We also have from (2.3) and (2.4)

$$g(F_rX,V)+g(t_rV,X)=0,\quad r=1,2,3, \tag{2.7}$$

which give the relations between $F_r$ and $t_r$.

Now, applying $\phi$, $\psi$ and $\theta$ to (2.3) respectively and using (2.1), (2.3) and (2.4), we can obtain

$$\begin{cases} P_1^2=-I+\alpha\otimes\xi-t_1F_1,\quad P_2^2=-I+\beta\otimes\eta-t_2F_2,\\ P_3^2=-I+\gamma\otimes\zeta-t_3F_3,\end{cases} \tag{2.8}$$

$$F_rP_r+f_rF_r=0,\quad r=1,2,3, \tag{2.9}$$

$$\begin{cases} P_2P_1=-P_3-t_2F_1+\alpha\otimes\eta,\quad P_3P_2=-P_1-t_3F_2+\beta\otimes\zeta,\\ P_1P_3=-P_2-t_1F_3+\gamma\otimes\xi,\\ P_1P_2=P_3-t_1F_2+\beta\otimes\xi,\qquad P_2p_3=P_1-t_2F_3+\gamma\otimes\eta,\\ P_3P_1=P_2-t_3F_1+\alpha\otimes\zeta,\end{cases} \tag{2.10}$$

and

$$(2.11)\quad \begin{cases} F_2P_1=-F_3-f_2F_1, & F_3P_2=-F_1-f_3F_2, \quad F_1P_3=-F_2-f_1F_3, \\ F_1P_2=F_3-f_1F_2, & F_2P_3=F_1-f_2F_3, \quad F_3P_1=F_2-f_3F_1. \end{cases}$$

Applying $\phi, \psi$ and $\theta$ to (2.4) respectively and using (2.1), (2.3) and (2.4), we also have

$$(2.12)\quad f_r^2=-I-F_rt_r,\quad r=1,2,3,$$

$$(2.13)\quad P_rt_r+t_rf_r=0,\quad r=1,2,3.$$

$$(2.14)\quad \begin{cases} f_2f_1=-f_3-F_2t_1, & f_3f_2=-f_1-F_3t_2, \quad f_1f_3=-f_2-F_1t_3, \\ f_1f_2=f_3-F_1t_2, & f_2f_3=f_1-F_2t_3, \quad f_3f_1=f_2-F_3t_1, \end{cases}$$

$$(2.15)\quad \begin{cases} P_2t_1+t_2f_1=-t_3, & P_3t_2+t_3f_2=-t_1, \quad P_1t_3+t_1f_3=-t_2, \\ P_1t_2+t_1f_2=t_3, & P_2t_3+t_2f_3=t_1, \quad P_3t_1+t_3f_1=t_2. \end{cases}$$

If we put $X=\xi, \eta, \zeta$ in (2.3) respectively and use (2.1), we have

$$(2.16)\quad P_1\xi=0,\quad P_2\eta=0,\quad P_3\zeta=0,$$

$$(2.17)\quad F_1\xi=0,\quad F_2\eta=0,\quad F_3\zeta=0,$$

$$(2.18)\quad \begin{cases} P_3\eta+F_3\eta=-P_2\zeta-F_2\zeta=-\xi, \quad P_1\zeta+F_1\zeta=-P_3\xi-F_3\xi=-\eta, \\ P_2\xi+F_2\xi=-P_1\eta-F_1\eta=-\zeta. \end{cases}$$

Finally, taking account of the last parts of (2.1), from (2.3) we have

$$(2.19)\quad \beta P_1=-\alpha P_2=-\gamma,\quad \gamma P_2=-\beta P_3=-\alpha,\quad \alpha P_3=-\gamma P_1=-\beta.$$

## 3. Contact-three-*CR* submanifolds of a manifold with Sasakian-three-structure

DEFINITION. Let $M$ be a submanifold isometrically immersed in a manifold $\bar{M}$ with Sasakian-three-structure $\{\xi, \eta, \zeta\}$ tangent to the structure vectors $\xi$, $\eta$ and $\zeta$. Then $M$ is called a contact-three-$CR$ submanifold of $\bar{M}$ if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_x(M)$ on $M$ satisfying the following conditions:

(1) $\mathcal{D}$ is invariant with respect to $\{\phi, \psi, \theta\}$, i.e.,

$$\phi\mathcal{D}_x \subset \mathcal{D}_x,\quad \psi\mathcal{D}_x \subset \mathcal{D}_x,\quad \theta\mathcal{D}_x \subset \mathcal{D}_x$$

for each $x \in M$, and

(2) the complementary orthogonal distribution $\mathcal{D}^\perp : x \to \mathcal{D}_x^\perp \subset T_x(M)$ is antiinvariant with respect to $\{\phi, \psi, \theta\}$, i.e.,

$$\phi\mathcal{D}_x^\perp \subset T_x(M)^\perp,\quad \psi\mathcal{D}_x^\perp \subset T_x(M)^\perp,\quad \theta\mathcal{D}_x^\perp \subset T_x(M)^\perp$$

for each $x \in M$.

REMARK. For a contact-three-$CR$ submanifold $\xi$, $\eta$ and $\zeta$ belong to $\mathcal{D}$. Indeed, from $\phi^2X=-X+\alpha(X)\xi$ for any $X \in \mathcal{D}$, we see that $\alpha(X)\xi \in \mathcal{D}$. Thus we have $\xi \in \mathcal{D}$ or $\alpha(X)=0$ and hence $\xi \in D^\perp$. When $\xi \in \mathcal{D}$, the condition (1) implies $\eta \in \mathcal{D}$ and $\zeta \in \mathcal{D}$ because $\theta\xi=-\eta$ and $\psi\xi=\zeta$. By the way the case that $\xi \in \mathcal{D}^\perp$ does not occur because of the condition (2). Hence

$\xi, \eta$ and $\zeta$ belong to $\mathcal{D}$.

Let $M$ be a contact-three $CR$ submanifold of a manifold $\bar{M}$ with a Sasakian-three-structure $\{\xi, \eta, \zeta\}$. We denote by $l$ and $l^{\perp}$ the projection operators on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Then we have

(3.1) $$l+l^{\perp}=I,\ \ l^2=l,\ \ l^{\perp 2}=l^{\perp},\ \ ll^{\perp}=l^{\perp}l=0.$$

From (2.3), we obtain

$$\phi lX=P_1lX+F_1lX,\ \ \psi lX=P_2lX+F_2lX,\ \ \theta lX=P_3lX+F_3lX,$$

which and the distribution $\mathcal{D}$ being invariant yield

(3.2) $$l^{\perp}P_rl=0,\ \ F_rl=0,\ \ r=1,2,3.$$

From (2.3), we also have

$$\phi l^{\perp}X=P_1l^{\perp}X+F_1l^{\perp}X,\ \ \psi l^{\perp}X=P_2l^{\perp}X+F_2l^{\perp}X,\ \ \theta l^{\perp}X=P_3l^{\perp}X+F_3l^{\perp}X,$$

from which, the distribution $\mathcal{D}^{\perp}$ being anti-invariant, we find

(3.3) $$P_rl^{\perp}=0,\ \ r=1,2,3$$

and consequently

(3.4) $$P_rl=P_r,\ \ r=1,2,3$$

because of $l^{\perp}=I-l$.

Now applying $l$ from the right to (2.9) and using (3.2) and (3.4), we find

(3.5) $$F_rP_r=0,\ \ r=1,2,3$$

and consequently

(3.6) $$f_rF_r=0,\ \ r=1,2,3.$$

Moteover, remembering the skew-symmetry of $f_r$ and the relation (2.7), we find

(3.7) $$t_rf_r=0,\ \ r=1,2,3$$

and consequently

(3.8) $$P_rt_r=0,\ \ r=1,2,3.$$

Thus, from (2.8) and (2.16) we have

(3.9) $$P_r^3+P_r=0,\ \ r=1,2,3,$$

which means that $\{P_1, P_2, P_3\}$ is an $f$-three-structure (see [4]) in $M$. Also, from (2.12) and (3.6) we have

(3.10) $$f_r^3+f_r=0,\ \ r=1,2,3,$$

which shows that $\{f_1, f_2, f_3\}$ is an $f$-three-structure in the normal bundle $T(M)^{\perp}$.

On the other hand, applying $l^{\perp}$ from the right to (2.10) and using (3.2) and (3.3), we can obtain

(3.11) $$\begin{cases} t_2F_1=0, & t_3F_2=0, & t_1F_3=0, \\ t_1F_2=0, & t_2F_3=0, & t_3F_1=0, \end{cases}$$

and consequently

(3.12) $$\begin{cases} P_2P_1=-P_3+\alpha\otimes\eta, & P_3P_2=-P_1+\beta\otimes\zeta, & P_1P_3=-P_2+\gamma\otimes\xi, \\ P_1P_2=P_3+\beta\otimes\xi, & P_2P_3=P_1+\gamma\otimes\eta, & P_3P_1=P_2+\alpha\otimes\zeta. \end{cases}$$

Consequently, for a submanifold $M$ of a manifold $\bar{M}$ with Sasakian-three-structure $\{\xi, \eta, \zeta\}$, assume that we have (3.5) and (3.11). Then we have (3.6)~(3.8) and consequently (3.9), (3.10) and (3.12).

Applying $F_2$ from the left to the first equation of (3.12) and using (2.17) and (3.5), we find

$$F_2P_3=0,$$

which and (2.11) give

$$F_1=f_2F_3$$

and consequently

$$F_1\zeta=0.$$

Similarly we have

(3.13) $$F_1\eta=0,\ F_1\zeta=0,\ F_2\xi=0,\ F_2\zeta=0,\ F_3\xi=0,\ F_3\eta=0,$$

from which and (2.18),

(3.14) $$P_1\eta=\zeta,\ P_1\zeta=-\eta,\ P_2\xi=-\zeta,\ P_2\zeta=\xi,\ P_3\xi=\eta,\ P_3\eta=-\xi.$$

Applying $P_3$ from the left to the first equation of (3.12) and making use of (2.19) and (3.14), we also have

$$-P_1^2+\alpha\otimes\xi=-P_3^2+\gamma\otimes\zeta.$$

Similarly we have

$$-P_1^2+\alpha\otimes\xi=-P_2^2+\beta\otimes\eta=-P_3^2+\gamma\otimes\zeta.$$

We now put

(3.15) $$l=-P_1^2+\alpha\otimes\xi,\ l^\perp=I-l.$$

Then we can easily verify that

$$l+l^\perp=I,\ l^2=l,\ l^{\perp 2}=l^\perp,\ ll^\perp=l^\perp l=0,$$

which means that $l$ and $l^\perp$ are complementary projection operators and consequently define complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively.

From the first equation of (3.15), we find

$$P_rl=P_r,\ r=1, 2, 3$$

because of (2.16) and (3.9). These equations can be written as

$$P_rl^\perp=0,\ r=1, 2, 3.$$

By the way $g(P_rX, Y)$ $(r=1, 2, 3)$ are skew-symmetric and $g(l^\perp X, Y)$ is symmetric and consequently the above equations yield

$$l^\perp P_r=0,\ r=1, 2, 3$$

and hence

$$l^\perp P_rl=0,\ r=1, 2, 3.$$

Also, from (2.17), (3.5) and (3.15), we find

$$F_rl=0,\quad r=1, 2, 3$$

and therefore

$$F_rl^\perp=F_r,\ r=1, 2, 3.$$

The above equations show that the distribution $\mathcal{D}$ is invariant and $\mathcal{D}^\perp$ is antiinvariant with respect to $\{\phi, \psi, \theta\}$. Moreover, we have

$$l\xi=\xi,\quad l\eta=\eta,\quad l\zeta=\zeta,$$
$$l^{\perp}\xi=0,\quad l^{\perp}n=0,\quad l^{\perp}\zeta=0$$

and consequently $\mathcal{D}$ contains $\xi, \eta$ and $\zeta$. Thus we have

THEOREM 1. *In order for a submanifold $M$ of a manifold with Sasakian-three-structure to be a contact-three-CR submanifold, it is necessary and sufficient that* (3.5) *and* (3.11) *are valid on $M$.*

THEOREM 2. *Let $M$ be a contact-three-CR submanifold of a manifold with Sasakian-three-structure. Then $\{P_1, P_2, P_3\}$ is an f-three-structure in $M$ and $\{f_1, f_2, f_3\}$ is an f-three-structure in the normal bundle.*

Let $M$ be a contact-three-$CR$ submanifold of a manifold $\bar{M}$ with a Sasakian-three-structure. If dim $\mathcal{D}=0$, then $M$ is called an anti-invariant submanifold of $\bar{M}$, and if dim $\mathcal{D}^{\perp}=0$, then $M$ is called an invariant submanifold of $\bar{M}$. If $\phi\mathcal{D}^{\perp}=T(M)^{\perp}$, $\psi\mathcal{D}^{\perp}=T(M)^{\perp}$ and $\theta\mathcal{D}^{\perp}=T(M)^{\perp}$, then $M$ is called a generic submanifold of $\bar{M}$.

## 4. Integrability of distributions

Let $\bar{M}$ be a $(4n+3)$-dimensional Riemannian manifold with a Sasakian-three-structure $\{\xi, \eta, \zeta\}$ and covered by a system of coordinate neighborhoods $\{\bar{U}\,;\, y^h\}$* we denote by $\{\xi^h, \eta^h, \zeta^h\}$ local components of $\{\xi, \eta, \zeta\}$ in $\bar{U}$ and put

$$\phi_i{}^h=\bar{\nabla}_i\xi^h,\quad \psi_i{}^h=\bar{\nabla}_i\eta^h,\quad \theta_i{}^h=\bar{\nabla}_i\zeta^h.$$

Then $\{\phi_i{}^h, \psi_i{}^h, \theta_i{}^h\}$ are local components of $\{\phi, \psi, \theta\}$ in $\bar{U}$.

Let $M$ be a contact-three-$CR$ submanifold of $\bar{M}$ and covered by a system of coordinate neighborhoods $\{U\,;\,x^a\}$. Let $M$ be represented by $y^h=y^h(x^a)$ with respect to local coordinates $(y^h)$ in $\bar{U}$ and $(x^a)$ in $U$. Denoting the vectors $\partial_a y^h (\partial_a=\partial/\partial x^a)$ tangent to $M$ by $B_a{}^h$ and unit normal vector fields by $C_x{}^h$, (2.3) and (2.4) can be written by

$$(4.1)\quad \begin{cases} \phi_i{}^h B_a{}^i=\phi_a{}^b B_b{}^h+\phi_a{}^x C_x{}^h, & \psi_i{}^h B_a{}^i=\psi_a{}^b B_b{}^h+\psi_a{}^x C_x{}^h,\\ \theta_i{}^h B_a{}^i=\theta_a{}^b B_b{}^h+\theta_a{}^x C_x{}^h, & \\ \phi_i{}^h C_x{}^i=-\phi_x{}^a B_a{}^h+\phi_x{}^y C_y{}^h, & \psi_i{}^h C_x{}^i=-\psi_x{}^a B_a{}^h+\psi_x{}^y C_y{}^h,\\ \theta_i{}^h C_x{}^i=-\theta_x{}^a B_a{}^h+\theta_x{}^y C_y{}^h, & \end{cases}$$

in each coordinate neighborhood $U$. Since $\{\xi, \eta, \zeta\}$ are tangent to $M$, we have in each $U$

$$(4.2)\qquad \xi^h=\xi^a B_a{}^h,\quad \eta^h=\eta^a B_a{}^h,\quad \zeta^h=\zeta^a B_a{}^h,$$

The indices $h, i, j, k$; $a, b, c, d, e$; $x, y, z, w$ run over the ranges $\{1, 2, \ldots, 4n+3\}$; $\{1, 2, \ldots, n+3\}$; $\{m+4, \cdots, 4n+3\}$ respectively and the summaticn convention will be used with respect to thoes systems of indices.

where $\xi^a$, $\eta^a$ and $\zeta^a$ are local vector fields defined in $U$. Denoting by $g_{ba}=g_{ji}B_b{}^jB_a{}^i$ and $g_{yx}=g_{ji}C_y{}^jC_x{}^i$, we can see that they are components of the induced metric tensors on $M$ and on the normal bundle since the immersion is isometric. If we put $\phi_{ab}=\phi_a{}^e g_{eb}$, $\phi_{bx}=\phi_b{}^y g_{yx}$, $\phi_{xb}=\phi_x{}^a g_{ab}$ and $\phi_{xy}=\phi_x{}^z g_{zy}$, it follows from (2. 5)$\sim$(2. 7) that

$$\text{(4. 3)}\qquad \begin{cases} \phi_{ab}=-\phi_{ba}, & \psi_{ab}=-\psi_{ba}, & \theta_{ab}=-\theta_{ba}, \\ \phi_{ax}=\phi_{xa}, & \psi_{ax}=\psi_{xa}, & \theta_{ax}=\theta_{xa}, \\ \phi_{xy}=-\phi_{yx}, & \psi_{xy}=-\psi_{yx}, & \theta_{xy}=-\theta_{yx} \end{cases}$$

Moreover, by means of Theorem 1, we have from (3. 5) and (3. 11)

$$\text{(4. 4)}\qquad \phi_e{}^x\phi_a{}^e=0,\quad \psi_e{}^x\psi_a{}^e=0,\quad \theta_e{}^x\theta_a{}^e=0,$$

$$\text{(4. 5)}\qquad \phi_a{}^x\psi_x{}^b=0,\quad \phi_a{}^x\theta_x{}^b=0,\quad \psi_a{}^x\theta_x{}^b=0,$$

and consequently, from (3. 7) and (3. 12),

$$\text{(4. 6)}\qquad \phi_a{}^y\phi_y{}^x=0,\quad \psi_a{}^y\psi_y{}^x=0,\quad \theta_a{}^y\theta_y{}^x=0,$$

$$\text{(4. 7)}\qquad \begin{cases} \phi_a{}^e\psi_e{}^b=-\theta_a{}^b+\alpha_a\eta^b, \quad \psi_a{}^e\theta_e{}^b=-\phi_a{}^b+\beta_a\zeta^b, \\ \theta_a{}^e\phi_e{}^b=-\psi_a{}^b+\gamma_a\xi^b, \end{cases}$$

where $\alpha_a$, $\beta_a$ and $\gamma_a$ are components of 1-forms $\alpha$, $\beta$ and $\gamma$ respectively. Also, from (3. 8), we have

$$\text{(4. 8)}\qquad \begin{cases} \phi_e{}^x\psi_a{}^e=0, \quad \phi_e{}^x\theta_a{}^e=0, \quad \psi_e{}^x\phi_a{}^e=0, \quad \psi_e{}^x\theta_a{}^e=0, \\ \theta_e{}^x\phi_a{}^e=0, \quad \theta_e{}^x\psi_a{}^e=0 \end{cases}$$

which and (2. 15) imply

$$\text{(4. 9)}\qquad \phi_a{}^x=\psi_a{}^z\theta_z{}^x,\quad \psi_a{}^x=\theta_a{}^z\phi_z{}^x,\quad \theta_a{}^x=\phi_a{}^z\psi_z{}^x.$$

From now on we denote by $\nabla_b$ the operator of covariant differentiation induced on $M$ from that of $\bar{M}$. Then the equations of Gauss and Weingarten are respectively given by

$$\text{(4. 10)}\qquad \nabla_b B_a{}^h=A_{ba}{}^x C_x{}^h,\quad \nabla_b C_x{}^h=-A_b{}^a{}_x B_a{}^h,$$

where $A_{ba}{}^x$ are the second fundamental tensors with respect to the unit normals $C_x{}^h$, $A_b{}^a{}_x=A_{be}{}^y g^{ea} g_{yx}$ and $(g^{ba})=(g_{ba})^{-1}$.

When $A_{ba}{}^x=0$ for all indices, it is said that $M$ is *totally geodesic*. When

$$\text{(4. 11)}\qquad A_{be}{}^x A_a{}^e{}_y-A_{ae}{}^x A_b{}^e{}_y=0$$

for all indices, we say that *the second fundamental tensors are commutative.*

Applying the operator $\nabla_c$ to (4. 1) and using (2. 2), (4. 1) and (4. 10), we can easily see that

$$\text{(4. 12)}\qquad \begin{cases} \nabla_c\phi_a{}^b=\alpha_a\delta_c{}^b-g_{ca}\xi^b+A_c{}^b{}_x\phi_a{}^x-A_{ca}{}^x\phi_x{}^b, \\ \nabla_c\psi_a{}^b=\beta_a\delta_c{}^b-g_{ca}\eta^b+A_c{}^b{}_x\psi_a{}^x-A_{ca}{}^x\psi_x{}^b \\ \nabla_c\theta_a{}^b=\gamma_a\delta_c{}^b-g_{ca}\zeta^b+A_c{}^b{}_x\theta_a{}^x-A_{ca}{}^x\theta_x{}^b \end{cases}$$

$$\text{(4. 13)}\qquad \begin{cases} \nabla_c\phi_a{}^x=A_{ca}{}^y\phi_y{}^x-A_{ce}{}^x\phi_a{}^e, \\ \nabla_c\psi_a{}^x=A_{ca}{}^y\psi_y{}^x-A_{ce}{}^x\psi_a{}^e, \\ \nabla_c\theta_a{}^x=A_{ca}{}^y\theta_y{}^x-A_{ce}{}^x\theta_a{}^e \end{cases}$$

and

$$
(4.14) \qquad \begin{cases} \nabla_c \phi_y{}^x = A_{ce}{}^x \phi_y{}^e - A_c{}^e{}_y \phi_e{}^x, \\ \nabla_c \psi_y{}^x = A_{ce}{}^x \psi_y{}^e - A_c{}^e{}_y \psi_e{}^x, \\ \nabla_c \theta_y{}^x = A_{ce}{}^x \theta_y{}^e - A_c{}^e{}_y \theta_e{}^x. \end{cases}
$$

Applying the operator $\nabla_c$ to (4.2) and using (4.10), wecan also obtain

$$
(4.15) \qquad \nabla_c \xi^a = \phi_c{}^a, \quad \nabla_c \eta^a = \psi_c{}^a, \quad \nabla_c \zeta^a = \theta_c{}^a
$$

and

$$
(4.16) \qquad A_{ce}{}^x \xi^e = \phi_c{}^x, \quad A_{ce}{}^x \eta^e = \psi_c{}^x, \quad A_{ce}{}^x \zeta^e = \theta_c{}^x.
$$

Now we consider the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$.

Let $X, Y \in \mathcal{D}^\perp$. Denoting by $[X, Y]^a$ local components of $[X, Y]$, we have

$$
(4.17) \qquad \phi_i{}^h [X, Y]^a B_a{}^i = Y^b X^a (\nabla_b \phi_a{}^e - \nabla_a \phi_b{}^e) B_e{}^h + [X, Y]^a \phi_a{}^x C_x{}^h
$$

because of $\phi_a{}^b X^a = 0$. By the way, applying the operator $\nabla_c$ to $\phi_{ba} X^b Y^a = 0$, we can see that

$$
(4.18) \qquad (A_{ca}{}^x \phi_{bx} - A_{cb}{}^x \phi_{ax}) X^b Y^a = 0
$$

because of (4.12), $\alpha_a X^a = 0$ and $\alpha_a Y^a = 0$.

Thus, substituting (4.12) into (4.17) and using (4.18), we find

$$
\phi_i{}^h [X, Y]^a B_a{}^i = [X, Y]^a \phi_a{}^x C_x{}^h.
$$

Similarl we have

$$
\psi_i{}^h [X, Y]^a B_a{}^i = [X, Y]^a \psi_a{}^x C_x{}^h,
$$
$$
\theta_i{}^h [X, Y]^a B_a{}^i = [X, Y]^a \theta_a{}^x C_x{}^h
$$

for any $X, Y \in \mathcal{D}^\perp$.

Thus we have

THEOREM 3. *Let $M$ be an $(m+3)$-dimensional contact-three-CR submanifold of a $(4n+3)$-dimensional manifold $\bar{M}$ with a Sasakian-three-structure $\{\xi, \eta, \zeta\}$. Then the distribution $\mathcal{D}$ is completely integrable and its maximal integral submanifold is a $p$-dimensional anti-invariant submanifold of $\bar{M}$ normal to $\xi, \eta$ and $\zeta$.*

Let $X, Y \in \mathcal{D}$. Then it follows that

$$
(4.19) \qquad \phi_b{}^x X^b = \phi_b{}^x Y^b = 0, \quad \psi_b{}^x X^b = \psi_b{}^x Y^b = 0, \quad \theta_b{}^x X^b = \theta_b{}^x Y^b = 0.
$$

Thus we have

$$
\phi_i{}^h [X, Y]^a B_a{}^i = [X, Y]^a \phi_a{}^b B_b{}^h + Y^b X^a (\nabla_b \phi_a{}^x - \nabla_a \phi_b{}^x) C_x{}^h,
$$

from which, substituting (4.13)

$$
\phi_i{}^h [X, Y]^a B_a{}^i = [X, Y]^a \phi_a{}^b B_b{}^h + (A_{be}{}^x \phi_a{}^e - A_{ae}{}^x \phi_b{}^e) X^b Y^a C_x{}^h.
$$

Similarly we have

$$
\psi_i{}^h [X, Y]^a B_a{}^i = [X, Y]^a \psi_a{}^b B_b{}^h + (A_{be}{}^x \psi_a{}^e - A_{ae}{}^x \psi_b{}^e) X^b Y^a C_x{}^h,
$$
$$
\theta_i{}^h [X, Y]^a B_a{}^i = [X, Y]^a \theta_a{}^b B_b{}^h + (A_{be}{}^x \theta_a{}^e - A_{ae}{}^x \theta_b{}^e) X^b Y^a C_x{}^h
$$

for any $X, Y \in \mathcal{D}$.

Hence we have

THEOREM 4. *Let $M$ be an $(m+3)$-dimensional contact-three-CR submanifold of a $(4n+3)$-dimensional manifold $\bar{M}$ with a Sasakian-three-structure $\{\xi, \eta, \zeta\}$ Then the distribution $\mathcal{D}$ is completely integrable if and only if*

$$(A_{be}{}^{x}\phi_{a}{}^{e}-A_{ae}{}^{x}\phi_{b}{}^{e})X^{b}Y^{a}=0, \quad (A_{be}{}^{x}\psi_{a}{}^{e}-A_{ae}{}^{x}\psi_{b}{}^{e})X^{b}Y^{a}=0,$$

$$(A_{be}{}^{x}\theta_{a}{}^{e}-A_{ae}{}^{x}\theta_{b}{}^{e})X^{b}Y^{a}=0$$

*for any vector fields $X, Y\in\mathcal{D}$. The maximal integral submanifold of $\mathcal{D}$ is an $(n+3-p)$-dimensional invariant submanifold of $\bar{M}$.*

## 5. Contact-three-CR submanifolds with commutative second fundamental tensors

Let $M$ be a contact-three-$CR$ submanifold of a manifold $\bar{M}$ with a Sasakian-three-structure $\{\xi, \eta, \zeta\}$. Assume that the second fundamental tensors are commutative, i. e., that (4. 11) is valid at each point of $M$. Transvecting (4. 11) with $\xi^{a}, \eta^{a}$ and $\zeta^{a}$ respectively, and using (4. 16), we have

(5. 1) $$A_{be}{}^{x}\phi_{y}{}^{e}=A_{b}{}^{e}{}_{y}\phi_{e}{}^{x}, \quad A_{be}{}^{x}\psi_{y}{}^{e}=A_{b}{}^{e}{}_{y}\psi_{e}{}^{x}, \quad A_{be}{}^{x}\theta_{y}{}^{e}=A_{b}{}^{e}{}_{y}\theta_{e}{}^{x}.$$

Now, applying the operator $\nabla_{b}$ to the first equation of (4. 6), we have

$$(\nabla_{b}\phi_{a}{}^{y})\phi_{y}{}^{x}+\phi_{a}{}^{y}\nabla_{b}\phi_{y}{}^{x}=0,$$

from which, substituting (4. 13) and (4. 14) and using (5. 1),

$$A_{ba}{}^{z}\phi_{z}{}^{y}\phi_{y}{}^{x}-A_{be}{}^{y}\phi_{a}{}^{e}\phi_{y}{}^{x}=0.$$

Transvecting this equation with $\phi_{c}{}^{a}$ and using (2. 8), we can easily see that

$$A_{bc}{}^{y}\phi_{y}{}^{x}=-A_{be}{}^{z}\phi_{z}{}^{y}\phi_{y}{}^{x}\phi_{c}{}^{e}$$

because of $A_{be}{}^{y}\phi_{y}{}^{x}\xi^{e}=0$ and $A_{be}{}^{y}\phi_{y}{}^{x}\phi_{z}{}^{e}=0$. Thus we have

$$\|A_{bc}{}^{y}\phi_{y}{}^{x}\|^{2}=(A_{bc}{}^{y}\phi_{y}{}^{x})(A^{bcz}\phi_{zx})=-(A_{be}{}^{w}\phi_{w}{}^{y}\phi_{y}{}^{x}\phi_{c}{}^{e})(A^{bcz}\phi_{zx})=0$$

since $A_{be}{}^{w}A_{c}{}^{e}{}_{z}$ is symmetric with respect to $b$ and $c$ and $\phi^{bc}$ is skew-symmetric. Hence we have $A_{bc}{}^{y}\phi_{y}{}^{x}=0$. Similarly we have

(5. 2) $$A_{bc}{}^{y}\phi_{y}{}^{x}=0, \quad A_{bc}{}^{y}\psi_{y}{}^{x}=0, \quad A_{bc}{}^{y}\theta_{y}{}^{x}=0,$$

which and (4. 9) imply

(5. 3) $$A_{bc}{}^{y}\phi_{y}{}^{a}=0, \quad A_{bc}{}^{y}\psi_{y}{}^{a}=0, \quad A_{bc}{}^{y}\theta_{y}{}^{a}=0.$$

Transvecting the first equation of (5. 3) with $\phi_{a}{}^{x}$ and using (2. 12) and (5. 2), we find

$$A_{bc}{}^{x}=0.$$

Thus we have

THEOREM 5. *Let $M$ be a contact-three-$CR$ submanifold of a manifold with a Sasakian-three-structure. If the second fundamental tensors are commutative, then $M$ is totally geodesic.*

## References


1. M. Barros, B. Y. Chen and F. Urbano, *Quaternion CR-submanifolds of quaternion manifolds*, Kodai Math. J. **4** (1981), 399-417.
2. A Bejancu, *CR submanifolds of a Kaehler manifold* I, Proc. Amer. Math. Soc. **69** (1978), 135-142.
3. B. Y. Chen, *On CR-submanifolds of a Kaehler manifold*, to appear.
4. S. Ishihara and M. Konishi, *On f-three-structures*, Hokkaido Math. J. **1** (1972), 127-135.
5. _______ *Fibred Riemannian spaces with Sasakian 3-structure*, Differential Geometry in Honor of K. Yano, Kinokuniya, Tokyo, 1972, 179-194.
6. Y. Y. Kuo, *On almost contact 3-structure*, Tôhoku Math. J. **22** (1970), 325-332.
7. Y. Y. Kuo and S. Tachibana. *On the distribution appeared in contact 3-structure*, Kaita J. Math. **2** (1970), 17-24.
8. J. S. Pak, *Quaternionic CR-submanifolds of quaternionic space forms*, to appear.
9. S. Sasaki, *Almost contact manifolds* I, Lecture note, Tôhoku Univ., 1965.
10. K. Yano and M. Kon, *Differential geometry of CR-submanifolds*, to appear in Geometriae Dedicata.
11. _______, *Contact CR submanifolds*, to appear in Kodal Math. J.


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