J. Korean Math. Soc.
Vol. 18, No. 2, 1982

ON THE DERIVED NORMAL RINGS OF A NOETHERIAN INTEGRAL DOMAIN

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1. It is well known that the derived normal ring of a noetherian integral domain is a Krull domain, which is a theorem of Mori-Nagata. J. Nishimura proved the theorem using properties of a complete local ring ([5]) and J. Querrè proved the same theorem without using either completion or properties of a complete local ring ([7]). In this paper we obtain some properties of an ideal transform and the derived normal ring of a noetherian integral domain, which give another proof of the theorem.

2. In this paper all rings are commutative with identity and a local ring $(A, \mathfrak{m})$ means a noetherian commutative ring A with only one maximal ideal $\mathfrak{m}$. We denote the Krull dimension of a ring A by $\dim A$.

PROPOSITION 1. ([3], (33.2)) *Let A be a noetherian integral domain with field of quotients K, Let L be a finite algebraic extension of K, and let B be a ring between A and L. If* $\dim A=1$, *then B is noetherian with* $\dim B\leqq 1$ *and for any non-zero element a in A, B/aB is a finite A-module.*

Proof. If $\dim A=1$, it is well-known Krull-Akizuki's theorem and so we omit the proof.

Let I be an ideal of a ring A, and let $Q(A)$ be the total quotient ring of A. $A(I)=\{x\in Q(A) : xI^n\subset A$ for some natural number $n\}$ is called the ideal transform of A with respect to I. We denote the derived normal ring of an integral domain A by $\bar{A}$, which is the integral closure of A in its field of quotients.

PROPOSITION 2. *Let $(A, \mathfrak{m})$ be a local domain and* $\dim A\geqq 2$. *Then the following conditions are equivalent*: (1) $A(\mathfrak{m})=A$, (2) $A(\mathfrak{m})\cap\bar{A}=A$.

Proof. Since $\dim A\geqq 2$, we have $x\mathfrak{m}\subset\mathfrak{m}$ for $x\in A:\mathfrak{m}$, and it follows that $A(\mathfrak{m})\neq A$ implies $A(\mathfrak{m})\cap\bar{A}\neq A$.

PROPOSITION 3. *Let $(A, \mathfrak{m})$ be a local domain and* $\dim A=n\geqq 2$. *Suppose for any noetherian integral domain B with* $\dim B<n$, *$\bar{B}$ is a Krull domain.*

Received October 20, 1981.

Then the following conditions are equivalent:
(1) $A(\mathfrak{m})\subset\bar{A}$, (2) $\bar{A}$ *has no maximal ideal of height* 1.

Proof. In view of Proposition (1.2) in [6] it follows that (2) implies (1). Now we shall prove (1) implies (2). By definition $A(\mathfrak{m})=\bigcap_{i=1}^{r}A_{f_i}$, where $\mathfrak{m}=f_1A+\cdots+f_rA$. Since $\dim A_{f_i}<\dim A=n$, by hypothesis $\bar{A}_{f_i}$, $i=1,\ldots,r$, is a Krull domain. Hence $\bar{A}=\overline{A(\mathfrak{m})}=\bigcap_{i=1}^{r}\bar{A}_{f_i}$ is a Krull domain.

Now suppose there is a maximal ideal $\bar{\mathfrak{m}}$ of $\bar{A}$ with ht $\bar{\mathfrak{m}}=1$. Then for non-zero element a in $\mathfrak{m}$, $a\bar{A}=\bar{\mathfrak{m}}^{(e)}\cap\bar{\mathfrak{p}}_2^{(e_2)}\cap\ldots\cap\bar{\mathfrak{p}}_n^{(e_n)}$. Since $\bar{\mathfrak{m}}^{(e)}+I=\bar{A}$ with $I=\bar{\mathfrak{p}}_2^{(e_2)}\cap\ldots\cap\bar{\mathfrak{p}}_n^{(e_n)}$, we have $b+c=1$ for some $b\in\bar{\mathfrak{m}}^{(e)}$ and $c\in I$.

Suppose there exist two distinct maximal ideals $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{n}}$ in $\bar{A}$ such that $\mathfrak{m}'=\bar{\mathfrak{m}}\cap A[b]=\bar{\mathfrak{n}}\cap A[b]$. Then, as $I\subset\bar{\mathfrak{n}}$, $c\in\bar{\mathfrak{n}}$ and $b\notin\bar{\mathfrak{n}}$. On the other hand $b\in\bar{\mathfrak{m}}^{(e)}\cap A[b]\subset\bar{\mathfrak{n}}$, which is a contradiction. Hence $\bar{\mathfrak{m}}$ is the only one maximal ideal of $\bar{A}$ which lies over $\mathfrak{m}'$.

Let $A_1=A[b]$, rad $A_1=\tilde{\mathfrak{m}}$, and $\mathfrak{m}'=\bar{\mathfrak{m}}\cap A_1$. Then, since $A(\mathfrak{m})$ is noetherian ([1]), $A_1(\tilde{\mathfrak{m}})$ is a finite $A(\mathfrak{m})$-module, and $A_1(\tilde{\mathfrak{m}})\subset\overline{A(\mathfrak{m})}=\bar{A}=\bar{A}_1$. Hence $\bar{A}_{\bar{\mathfrak{m}}}\supset A_1(\tilde{\mathfrak{m}})_{\mathfrak{m}'}=(A_1)_{\mathfrak{m}'}(\mathfrak{m}')=K$ because $\dim (A_1)_{\mathfrak{m}'}=1$. This is a contradiction. Therefore $\bar{A}$ has no maximal ideal of height 1.

PROPOSITION 4. *Let* $(A,\mathfrak{m})$ *be a local domain and* $B=A(\mathfrak{m})\cap\bar{A}$. *Then the set of maximal ideals of* $\bar{A}$ *with height* 1 *and the set of maximal ideals of* B *with height* 1 *are in one to one correspondence.*

Proof. $(B,\mathfrak{n}_1,\ldots,\mathfrak{n}_r)$ is a semi-local domain with Jacobson radical $\mathfrak{n}=\mathfrak{n}_1\ldots\mathfrak{n}_r$ ([1]). Suppose that $\mathfrak{n}_1,\ldots,\mathfrak{n}_\alpha$ are maximal ideals of height 1 and that $\mathfrak{n}_{\alpha+1},\ldots,\mathfrak{n}_r$ are maximal ideals of height>1. Then $B(\mathfrak{n})=\bigcap_{j>\alpha}B_{\mathfrak{n}_j}(\mathfrak{n}_j)=B_T(\mathfrak{n}_T)$, where $T=B-\bigcup_{j>\alpha}\mathfrak{n}_j$. Since $B(\mathfrak{n})\cap\bar{B}=A(\mathfrak{m})\cap\bar{A}=B$, $B_{\mathfrak{n}_j}=(B(\mathfrak{n})\cap\bar{B})_{\mathfrak{n}_j}=B_{\mathfrak{n}_j}(\mathfrak{n}_j)\cap\bar{B}_{\mathfrak{n}_j}$ for $j>\alpha$. Hence by Proposition 2, $B_{\mathfrak{n}_j}=B_{\mathfrak{n}_j}(\mathfrak{n}_j)$ for $j>\alpha$ and $B_T(\mathfrak{n}_T)=B(\mathfrak{n})_T=B_T\subset\bar{B}_T$. Thus by Proposition 3, $\bar{B}_T$ has no maximal ideal of height 1. Therefore if $\bar{\mathfrak{m}}$ is a maximal ideal of $\bar{A}$ with ht$\bar{\mathfrak{m}}=1$ and if $\mathfrak{n}=\bar{\mathfrak{m}}\cap B$, then ht$\mathfrak{n}=1$.

Now let $S_i=B-\mathfrak{n}_i$ for $i\leqq\alpha$. Then $B_{S_i}=(A(\mathfrak{m})\cap\bar{A})_{S_i}=A(\mathfrak{m})_{S_i}\cap\bar{A}_{S_i}=\bar{A}_{S_i}$. Hence $\bar{\mathfrak{m}}=\mathfrak{n}_iB_{S_i}\cap\bar{A}$ is the only maximal ideal of $\bar{A}$ which lies $\mathfrak{n}_i$.

PROPOSITION 5. *Let* $(A,\mathfrak{m})$ *be a local domain with* $\dim A=n\geqq2$. *Suppose that* $\bar{R}$ *is a Krull domain for any noetherian integral domain* R *of dimension* $<n$. *Then* $\bar{A}$ *is a Krull domain.*

Proof. Let $B=A(\mathfrak{m})\cap\bar{A}$. Then $(B,\mathfrak{n}_1,\ldots,\mathfrak{n}_r)$ is a semi-local domain. Let $\mathfrak{n}_i$, $1\leqq i\leqq\alpha$, be maximal ideals of height 1, and let $\mathfrak{n}_j$, $j>\alpha$, be maximal ideals of height>1. Then $\bar{B}=(\bigcap_{i=1}^{\alpha}\bar{B}_{\mathfrak{n}_i})\cap(\bigcap_{j=\alpha+1}^{r}\bar{B}_{\mathfrak{n}_j})$, where $\bar{B}_{\mathfrak{n}_i}$, $1\leqq i\leqq\alpha$, is a discrete valuation ring by Proposition 4. For $j>\alpha$, $B_{\mathfrak{n}_j}(\mathfrak{n}_j)\subset\bar{B}_{\mathfrak{n}_j}$ and $B_{\mathfrak{n}_j}(\mathfrak{n}_j)=\bigcap_{k=1}^{e_j}(B_{\mathfrak{n}_j})_{b_{jk}}$, where $\mathfrak{n}_j=(b_{j1},\dots,b_{je_j})$. Hence $\bar{B}_{\mathfrak{n}_j}=\overline{B_{\mathfrak{n}_j}(\mathfrak{n}_j)}=\bigcap_{k=1}^{e_j}\overline{(B_{\mathfrak{n}_j})}_{b_{jk}}$. Since $\dim\overline{(B_{\mathfrak{n}_j})}_{b_{jk}}\leqq n-1$, by hypothesis $\overline{(B_{\mathfrak{n}_j})}_{b_{jk}}$ is a Krull domain and hence $\bar{A}=\bar{B}$ is a Krull domain.

PROPOSITION 6. *Let A be a noetherian integral domain with* $\dim A=n\geqq 2$. *Suppose that $\bar{R}$ is a Krull domain for any local domain R with* $\dim R\leqq n$. *Then $\bar{A}$ is a Krull domain.*

Proof. $\bar{A}=\bigcap_{\mathfrak{m}}\bar{A}_{\mathfrak{m}}$, where the intersection runs through all maximal ideals of A. By hypothesis each $\bar{A}_{\mathfrak{m}}$ is a Krull domain, so it is sufficient to show that for any non-zero element a in A, $\mathfrak{J}=\{\bar{\mathfrak{p}}\in\mathrm{Spec}(\bar{A}):\ \mathrm{ht}\,\bar{\mathfrak{p}}=1,\ a\in\bar{\mathfrak{p}}\}$ is a finite set. Thus we shall prove the following:

(1) $\mathfrak{J}_0=\{\mathfrak{p}:\bar{\mathfrak{p}}\cap A=\mathfrak{p},\ \bar{\mathfrak{p}}\in\mathfrak{J}\}$ is finite.

(2) For each $\mathfrak{p}\in\mathfrak{J}_0$, $\mathfrak{J}_{\mathfrak{p}}=\{\bar{\mathfrak{p}}\in\mathfrak{J}:\bar{\mathfrak{p}}\cap A=\mathfrak{p}\}$ is finite.

Proof of (2). The number of elements in $\mathfrak{J}_{\mathfrak{p}}$ is the number of maximal ideals in $\bar{A}_{\mathfrak{p}}$ with height 1 which lies over the maximal ideal of the local domain $A_{\mathfrak{p}}$. Let $B=A_{\mathfrak{p}}(\mathfrak{p})\cap\bar{A}_{\mathfrak{p}}$. Then B is a semi-local domain ([1]). Hence $\mathfrak{J}_{\mathfrak{p}}$ is a finite set by Proposition 4.

Proof of (1). Let $\mathfrak{p}=\bar{\mathfrak{p}}\cap A$, $\bar{\mathfrak{p}}\in\mathfrak{J}$. If $\mathrm{ht}\,\mathfrak{p}=1$, then $\mathfrak{p}\in\mathrm{Ass}(aA)$. Thus the number of $\mathfrak{p}$ with $\mathrm{ht}\,\mathfrak{p}=1$ is finite. Now suppose $\mathrm{ht}\,\mathfrak{p}>1$. If $A_{\mathfrak{p}}(\mathfrak{p})=A_{\mathfrak{p}}$, then $A_{\mathfrak{p}}(\mathfrak{p})\subset\bar{A}_{\mathfrak{p}}$. Hence $\bar{A}_{\mathfrak{p}}$ has no maximal ideal of height 1 by Proposition 3. This is a contradiction. Therefore $A_{\mathfrak{p}}(\mathfrak{p})\supsetneqq A_{\mathfrak{p}}$. Now we need a lemma.

LEMMA. *Let $(A,\mathfrak{m})$ be a local domain with field of quotients K. If $A(\mathfrak{m})\supsetneqq A$ then for any $0\neq b$ in $\mathfrak{m}$, $\mathfrak{m}$ is an associated prime divisor of bA.*

Proof of Lemma. Let $\mathfrak{m}=(a_1,\dots,a_r)=(a_1,\dots,a_r,b)$. Then $A(\mathfrak{m})=\bigcap_{j=1}^{r}A_{a_j}\cap A_b\supsetneqq A$. Hence there is an element x in $A(\mathfrak{m})$ but $x\notin A$. We may assume $x\mathfrak{m}\subset A$. Write $x=d/b^i$. Then, as $x\mathfrak{m}\subset A$, $d\mathfrak{m}\subset b^iA$ i.e., $b^i:d\supset\mathfrak{m}$.

If $b^i:d\neq\mathfrak{m}$, then $b^i:d=A$ and $d=b^ia$ for, some a in A. Hence $x\in A$, which is a contradiction. Thus $b^i:d=\mathfrak{m}$. Now consider $b^{i-1}:d\supset\mathfrak{m}$. Then it follows that either there is an element c in A such that $b:c=\mathfrak{m}$ or $b^{i-1}:d=\mathfrak{m}$. Consequently, we have $b:c=\mathfrak{m}$ for some c in A. Lemma is proved.

By Lemma $\mathfrak{p}$ is an associated prime ideal of aA but the number of associated prime ideals of aA is finite. Hence $\mathfrak{J}_0$ is finite.

PROPOSITION 7. ([3], (33.10)) *The derived normal ring $\bar{A}$ of a noetherian integral domain A is a Krull domain.*

Proof. In view of Propositions 1, 5 and 6 the assertion is true for any noetherian domain of finite Krull dimension.

For a general noetherian domain A, since $\dim A_{\mathfrak{m}}<\infty$ for any maximal ideal $\mathfrak{m}$ and $\bar{A}=\bigcap_{\mathfrak{m}}\bar{A}_{\mathfrak{m}}$, to get the assertion, it is sufficient to show that for any $0\neq a$ in $\bar{A}$, $\mathcal{F}=\{\bar{\mathfrak{p}}\in \mathrm{Spec}(\bar{A}) : \mathrm{ht}\,\bar{\mathfrak{p}}=1,\ a\in\bar{\mathfrak{p}}\}$ is a finite set. But the same reasoning as in Proposition 6 gives the claim.

PROPOSITION 8. ([3], (33.12)) *The derived normal ring $\bar{A}$ of a noetherian integral domain A of Krull dimension $\leqq 2$ is again noetherian.*

Proof. If $\dim A\leqq 1$, the assertion is clear by Proposition 1. Suppose $\dim A=2$. Let $\bar{\mathfrak{p}}$ be a prime ideal of $\bar{A}$ with height 1 and let $\mathfrak{p}=\bar{\mathfrak{p}}\cap A$. If $\bar{\mathfrak{p}}$ is not maximal, then $\mathrm{ht}\,\mathfrak{p}=1$. Since the quotient field $\kappa(\bar{\mathfrak{p}})$ of $\bar{A}/\bar{\mathfrak{p}}$ is a finite algebraic extension of the quotient field $\kappa(\mathfrak{p})$ of $A/\mathfrak{p}$ and $\dim (A/\mathfrak{p})=1$, by Proposition 1 $\bar{A}/\bar{\mathfrak{p}}$ is noetherian. Thus by Mori–Nishimura's theorem ([4], Theorem), $\bar{A}$ is noetherian.

Note that in Proposition 1 B is not necessarily finite ([3], p. 205, Example 3). In Proposition 8 if B is a ring between A and $\bar{A}$, then B is not necessarily noetherian ([3], p. 207, Example 4), and in Proposition 7 even A is not necessarily noetherian ([3], p. 207, Example 5).

The author expresses his hearty thanks to Professor M. Nagata and Dr. J. Nishimura for their valuable suggestions.

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