

## REIDEMEISTER NUMBERS AND APPLICATIONS TO THE FIXED POINT THEORY

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### 1. Introduction

Let  $f : X \rightarrow X$  be a continuous map on a compact, connected, ANR  $X$ . In the study of fixed point theorems for a map  $f$ , several interesting numbers, namely the Lefschetz number  $L(f)$ , the Nielsen number  $N(f)$ , and the fixed point index  $i(f, X, U)$  play important roles. In many cases, the Nielsen number gives more complete geometric information about fixed points than other numbers. However, the Nielsen number is hard to compute in general. In an effort to compute the Nielsen number, Jiang was able to relate the number to the Reidemeister number of the induced homomorphism  $f_*$  on the fundamental group  $\Pi_1(X)$  where the space  $X$  is an  $H$ -space or a lens space ([1], [9]). We note that the fundamental groups of these spaces are abelian, and the Reidemeister number of a homomorphism from abelian group into itself is readily computable in many cases. In the case when the fundamental group of the space  $X$  is not abelian, almost nothing is known how to compute the Nielsen number except a few isolated cases ([1], [8]). In the case when the space  $X$  is a compact, connected, aspherical polyhedron, if a map  $f : X \rightarrow X$  has non-zero Lefschetz number and the induced homomorphism  $f_* : \Pi_1(X) \rightarrow \Pi_1(X)$  sends the fundamental group  $\Pi_1(X)$  into its center  $Z(\Pi_1(X))$ , then the Nielsen number of  $f$ ,  $N(f)$ , and the Reidemeister number of  $f_*$  on  $\Pi_1(X)$ ,  $R(f)$ , are the same. This theorem is implicitly stated in [1 : ch. VII].

In this paper, we find a formula for  $R(f)$  without the asphericity condition on the space, that is, the Reidemeister number of  $f$  is equal to that of  $f_*$  restricted to the center of  $\Pi_1(X)$ . This is used to show a similar theorem to the above theorem of Brown. We also apply the Reidemeister number to prove a product theorem for the Nielsen numbers of a fiber-preserving map triple  $(f, \bar{f}, f_b)$  on an orientable Hurewicz fibering.

We first study the Reidemeister numbers of homomorphisms on abstract groups in Section 2. Then we relate the Reidemeister number and the Nielsen

number of a map  $f : X \rightarrow X$  in Section 3. In Section 4, we study a product theorem for the Nielsen numbers of a fiber-preserving map triple on a Hurewicz fibering.

## 2. Reidemeister numbers on abstract groups

Let  $h : G \rightarrow G$  be a self homomorphism on a group  $G$ . Two elements  $\alpha$  and  $\beta$  in  $G$  are said to be the  $h$ -equivalent,  $\alpha \stackrel{h}{\sim} \beta$ , if there exists an element  $\gamma \in G$  such that  $\alpha = \gamma \beta h(\gamma^{-1})$ . This is an equivalence relation on  $G$  and divides  $G$  into equivalence classes. The set of all equivalence classes of  $G$  is denoted by  $G'(h) = \{[\alpha]\}$ . The cardinality of this set  $G'(h)$  is called the Reidemeister number of  $h$  on  $G$ , and it is denoted by  $R(h)$ .

If  $G$  is an abelian group, then we define an addition in  $G'(h)$  by  $[\alpha] + [\beta] = [\alpha + \beta]$  for elements  $[\alpha]$  and  $[\beta]$  in  $G'(h)$ . This is a well-defined operation on  $G'(h)$  and makes  $G'(h)$  an abelian group. Let  $h' : H \rightarrow H$  be a homomorphism on an abelian group  $H$ . If  $i : H \rightarrow G$  is a homomorphism such that  $ih' = hi$ , then  $i(\alpha)$  and  $i(\beta)$  are  $h$ -equivalent whenever  $\alpha$  and  $\beta$  are  $h'$ -equivalent. Thus the homomorphism  $i : H \rightarrow G$  induces a mapping  $i^* : H'(h') \rightarrow G'(h)$ , which is defined by  $i^*([\alpha]) = [i(\alpha)]$ . It is easy to see that the mapping  $i^* : H'(h') \rightarrow G'(h)$  is a homomorphism.

**PROPOSITION 2.1.** *Let  $h : G \rightarrow G$ ,  $h' : H \rightarrow H$ , and  $i : H \rightarrow G$  be homomorphisms on abelian groups such that  $ih' = hi$ . If the homomorphism  $i$  is a monomorphism and  $h(G) \subset i(H)$ , then  $i^* : H'(h') \rightarrow G'(h)$  is a monomorphism.*

*Proof:* Let  $i^*([\alpha]) = i^*([\beta])$ ,  $[\alpha], [\beta] \in H'(h')$ . Then there exists an element  $\gamma \in G$  such that  $i(\alpha) = \gamma + i(\beta) - h(\gamma)$ . Since  $h(\gamma) \in i(H)$ , there exists an element  $\delta \in H$  such that  $h(\gamma) = i(\delta)$ . Therefore, we have  $i(\alpha) = \gamma + i(\beta) - i(\delta)$ . This implies that the element  $\gamma$  belongs to  $i(H)$ . Let  $\gamma = i(\eta)$  for an element  $\eta \in H$ . Then  $i(\alpha) = i(\eta) + i(\beta) - h(i(\eta)) = i(\eta) + i(\beta) - i(h'(\eta))$ . Since  $i$  is a monomorphism, we have  $\alpha = \eta + \beta - h'(\eta)$ ,  $\eta \in H$ , that is,  $[\alpha] = [\beta]$  in  $H'(h')$ .

We note that if  $G$  is not an abelian group, then Reidemeister classes  $G'(h)$  may not be a group. Furthermore, the cardinality of inverse images of elements in  $G'(h)$  under  $i^* : H'(h') \rightarrow G'(h)$  are not the same even if  $H$  is a subgroup of  $G$ ,  $i$  is the inclusion, and  $h(G) \subset H$ . The cardinality of  $i^{*-1}([\alpha])$ ,  $[\alpha] \in G'(h)$ , plays an important role in Section 3.

In the case when  $G$  is not abelian, the induced mapping  $i^* : H'(h') \rightarrow G'(h)$  is not a homomorphism. However, we show that if  $i : H \rightarrow G$  is a monomorphism from an abelian group  $H$ , then  $i^*$  is an injective mapping.

PROPOSITION 2.2. *Let  $h : G \rightarrow G$  be a homomorphism on a group  $G$  (not necessarily an abelian group), and let  $h' : H \rightarrow H$  be a homomorphism on an abelian group  $H$ . If  $i : H \rightarrow G$  is a monomorphism such that  $hi = ih'$ , and if  $h(G) \subset i(H)$ , then  $i^* : H'(h') \rightarrow G'(h)$  is an injective mapping.*

The proof is similar to the one of Proposition 2.1. If  $H$  is a subgroup of a group  $G$ , then the collection of all elements  $[\alpha]$  of  $G'(h)$  such that  $[\alpha] \cap H \neq \emptyset$ , that is,  $\alpha \overset{h}{\sim} \beta$  for an element  $\beta \in H$ , will be called the Reidemeister classes of  $h$  restricted to  $H$ , and it will be denoted by  $H'(h)$ . If  $H$  is an invariant subgroup of  $G$  under the homomorphism  $h : G \rightarrow G$ , that is,  $h(H) \subset H$ , then the cardinality of  $H'(h|H)$  is bigger than the cardinality of  $H'(h)$ . If  $G$  is not an abelian group, then  $H'(h)$  may not have a group structure. However, if  $h : G \rightarrow G$  sends the whole group  $G$  into the center  $Z$  of  $G$ , then the Reidemeister classes of  $h$  restricted to  $Z$ ,  $Z'(h)$ , is an abelian group.

PROPOSITION 2.3. *Let  $Z$  be the center of the group  $G$ , and let  $h : G \rightarrow G$  be a homomorphism such that  $h(G) \subset Z$ . Then  $Z'(h)$  form an abelian group and two sets  $Z'(h)$  and  $G'(h)$  are the same.*

*Proof:* We note that  $Z'(h) = \{[\alpha] \in G'(h) \mid \alpha \overset{h}{\sim} \beta \text{ for an element } \beta \in Z\}$ . Let  $[\alpha], [\beta]$  be elements in  $Z'(h)$ . We need to show that the operation defined by  $[\alpha][\beta] = [\alpha\beta]$  is well-defined and closed in  $Z'(h)$ . We know that  $\alpha \overset{h}{\sim} \alpha'$  and  $\beta \overset{h}{\sim} \beta'$  for elements  $\alpha', \beta' \in Z$ , that is, there exist elements  $\gamma_1, \gamma_2 \in G$  such that  $\alpha = \gamma_1 \alpha' h(\gamma_1^{-1})$ , and  $\beta = \gamma_2 \beta' h(\gamma_2^{-1})$ . Let  $\gamma_1 \gamma_2 = \gamma \in G$ . Since  $\alpha', \beta'$ , and  $h(\gamma_1^{-1})$  are elements of  $Z$ , we have  $\gamma \alpha' \beta' h(\gamma^{-1}) = \gamma_1 \gamma_2 \alpha' \beta' h((\gamma_1 \gamma_2)^{-1}) = \gamma_1 \gamma_2 \alpha' \beta' h(\gamma_2^{-1}) h(\gamma_1^{-1}) = \gamma_1 \alpha' \gamma_2 \beta' h(\gamma_2^{-1}) h(\gamma_1^{-1}) = \gamma_1 \alpha' \beta h(\gamma_1^{-1}) = \gamma_1 \alpha' h(\gamma_1^{-1}) \beta = \alpha \beta$ , that is,  $\alpha \beta \overset{h}{\sim} \alpha' \beta'$ . Thus  $[\alpha][\beta] = [\alpha\beta]$  is a well-defined operation and  $[\alpha][\beta] = [\alpha\beta] = [\alpha' \beta'] = [\alpha'] [\beta'] \in Z'(h)$ . Obviously, we have  $[\alpha][\beta] = [\beta][\alpha]$  since  $[\alpha' \beta'] = [\beta' \alpha']$ . Let  $[\alpha] \in Z'(h)$ . Then  $[\alpha \alpha^{-1}] = [e] = [\alpha^{-1} \alpha]$ . Therefore  $[\alpha^{-1}] = [\alpha]^{-1}$  in  $Z'(h)$ . This proves that  $Z'(h)$  is an abelian group. By the definition of  $Z'(h)$ , we have  $Z'(h) \subset G'(h)$ . On the other hand, if  $[\alpha] \in G'(h)$ , then  $\alpha = \alpha h(\alpha) h(\alpha^{-1})$ , that is,  $\alpha \overset{h}{\sim} h(\alpha)$  and  $h(\alpha) \in H$ . Thus  $[\alpha] \in Z'(h)$ .

COROLLARY 2.4. *Let  $h : G \rightarrow G$  be a homomorphism on a group  $G$ , and let  $H$  be an abelian subgroup of  $G$  such that  $h(G) \subset H$ . If  $h' = h|H$ , then  $i^* : H'(h') \rightarrow G'(h)$  is a bijective mapping. Furthermore, if  $G$  is also abelian, then  $i^*$  is an isomorphism.*

*Proof:* We know that  $\alpha = \alpha h(\alpha) h(\alpha^{-1})$  for any class  $[\alpha] \in G'(h)$ . Since  $h(\alpha) \in H$ ,  $[\alpha] \in H'(h)$ . Therefore  $G'(h) = H'(h)$ . By Proposition 2.2, we see that  $i^* : H'(h') \rightarrow G'(h) = H'(h)$  is injective. Since  $\alpha \overset{h'}{\sim} \beta$  is equivalent to

$\alpha \stackrel{h}{\sim} \beta$ ,  $i^* : H'(h') \rightarrow H'(h)$  is also surjective. If  $G$  is also abelian, then Proposition 2.1 implies that  $i^*$  is an isomorphism.

REMARK 2.5. Let  $H$  be a subgroup of a group  $G$ , and let  $h' = h|_H$  for a homomorphism  $h : G \rightarrow G$ . We consider the following equivalence relations on  $H$ :

- (1) Two elements  $\alpha, \beta \in H$  are equivalent if there exists an element  $\gamma \in G$  such that  $\alpha = \alpha\beta h(\gamma^{-1})$ .
- (2) Two elements  $\alpha, \beta \in H$  are equivalent if there exists an element  $\gamma \in H$  such that  $\alpha = \alpha\beta h(\gamma^{-1})$ .
- (3) Two elements  $\alpha, \beta \in H$  are equivalent if there exists an element  $\gamma \in H$  such that  $\alpha = \alpha\beta h'(\gamma)$  under the assumption  $h' = h|_H : H \rightarrow H$ .

We denote the cardinality of corresponding equivalence classes on  $H$  of type (i) by  $R_i(h)$ ,  $i=1, 2, 3$ . The type (1) is equivalent to the usual definition. Therefore, the equivalence classes of type (1) is  $H'(h)$ , the Reidemeister classes of  $h$  restricted to  $H$ . We have clearly that  $R_1(h) \leq R_2(h)$ , and  $R_2(h) = R_3(h)$  if  $H$  is invariant under  $h : G \rightarrow G$ , that is,  $h' = h|_H : H \rightarrow H$ . Thus, if  $h(G) \subset H$ , then  $R_3(h)$  (=cardinality of  $H'(h')$ ) is equal to the Reidemeister number of  $f$ ,  $R(h)$ , and  $R(h) = R_1(h)$  by Corollary 2.4. Thus, we have  $R(h) = R_1(h) = R_2(h) = R_3(h)$ .

### 3. Nielsen numbers and Reidemeister numbers of maps

Let  $X$  be a compact, connected ANR, and  $f : X \rightarrow X$  be a continuous map such that the Lefschetz number  $L(f)$  is not zero. We denote the set of fixed points of  $f$  by  $\Phi(f) = \{x \in X \mid f(x) = x\}$ . We say that two points  $x_0$  and  $x_1$  in  $\Phi(f)$ , are  $f$ -equivalent if there is a path  $c : I \rightarrow X$  such that  $c(0) = x_0$ ,  $c(1) = x_1$ , and the path  $fc : I \rightarrow X$  is homotopic to  $c$  relative to end points. The  $f$ -equivalence is an equivalence relation on  $\Phi(f)$ . The equivalence classes are called fixed point classes of  $f$ . It is known that the set of fixed points classes of a map  $f$  on a compact, connected, ANR space  $X$  is finite. We denote the set of fixed point classes of  $f$  by  $\Phi'(f) = \{F_1, \dots, F_n\}$ . We say that a fixed point class  $F_j$  of  $f$  is essential if the fixed point index  $i(F) = i(X, f, U_j) \neq 0$ . (see [1]). The Nielsen number of the map  $f$ ,  $N(f)$ , is defined to be the number of fixed point classes of  $f$  that are essential. A fixed point theorem with this number is that any continuous map  $g : X \rightarrow X$  homotopic to  $f$  has at least  $N(f)$  fixed points.

The map  $f : X \rightarrow X$  induces the homomorphism  $f_* : \Pi_1(X, x_0) \rightarrow \Pi_1(X, x_0)$ , where the base point  $x_0$  is taken in  $\Phi(f)$  and it will be omitted in the sequel. We define the Reidemeister number of  $f$ ,  $R(f)$ , to be the Reidemeister number of the induced homomorphism  $f_* : \Pi_1(X) \rightarrow \Pi_1(X)$ , that is,

$R(f)$  = the cardinality of  $(\Pi_1(X))'(f_#)$  in the notation of Section 2. Then we have  $N(f) \leq R(f)$ , [1]. Let  $\text{Map}(X, X)$  be the set of all continuous self maps on  $X$ . Define a map  $p : \text{Max}(X, X) \rightarrow X$  by  $p(g) = g(x_0)$ ,  $x_0 \in \Phi(f)$ , then  $p$  induces a homomorphism  $p_* : \Pi_1(\text{Max}(X, X), f) \rightarrow \Pi_1(X, x_0) = \Pi_1(X)$ . The Jiang subgroup  $T(f, x_0)$  of  $\Pi_1(X)$  is the image of the homomorphism  $p_*$ . As in Section 2,  $(\Pi_1(X))'(f_#)$  denotes the set of equivalence classes of  $\Pi_1(X)$  under  $f_#$ -equivalence, and  $T'(f_#)$  denotes the classes of  $(\Pi_1(X))'(f_#)$  which contain elements of  $T(f)$ . The Jiang number,  $J(f)$ , is defined to be the cardinality of  $T'(f_#)$ . Then we have  $J(f) \leq N(f)$ , [1]. If  $\Pi_1(X) = T(f)$ , then all the fixed point classes of  $f$  have the same index, and there is at least one essential fixed point class of  $f$  since  $L(f) \neq 0$ . Thus, in this case, the Nielsen number  $N(f)$  is equal to the cardinality of  $\Phi'(f)$ , and  $J(f) = N(f) = R(f)$ , [1].

In [1: Ch. VII], it is implicitly stated that if a map  $f : X \rightarrow X$  on a compact, connected, aspherical polyhedron has non-zero Lefschetz number, and if the induced homomorphism  $f_* : \Pi_1(X) \rightarrow \Pi_1(X)$  sends  $\Pi_1(X)$  into the center  $Z(\Pi_1(X))$  of  $\Pi_1(X)$ , then  $N(f) = R(f)$ . If we assume that  $f_*(\Pi_1(X)) \subset Z(\Pi_1(X))$ , then Propositions in Section 2 imply the following theorem without the asphericity condition on  $X$ .

**THEOREM 3.1.** *Let  $f : X \rightarrow X$  be a continuous map on a compact, connected, ANR  $X$  such that  $L(f) \neq 0$ . If  $f_*(\Pi_1(X)) \subset Z(\Pi_1(X))$ , then  $R(f) =$  the cardinality of  $(Z(\Pi_1(X)))'(f_*') = R(f_*')$ , where  $f_*' = f_*|_{Z(\Pi_1(X))} : Z(\Pi_1(X)) \rightarrow Z(\Pi_1(X))$ . Thus  $R(f) = R(f_*') = \text{order} \left( \frac{Z(\Pi_1(X))}{(1-f_*')(Z(\Pi_1(X)))} \right)$  since  $Z(\Pi_1(X))$  is an abelian group.*

**COROLLARY 3.2.** *Let  $X$  be a compact, connected, aspherical polyhedron, and  $f : X \rightarrow X$  be a continuous map such that  $L(f) \neq 0$ . If  $f_*(\Pi_1(X)) \subset Z(\Pi_1(X))$ , then  $R(f_*') = J(f) = N(f) = R(f)$ .*

*Proof:* Since  $X$  is aspherical and  $f_*(\Pi_1(X)) \subset Z(\Pi_1(X))$ , we have  $Z(\Pi_1(X)) \subset T(f) = \Pi_1(X)$ , [1 : p. 103]. Thus we have  $(Z(\Pi_1(X)))'(f_*) \subset T'(f_*) = (\Pi_1(X))'(f_*)$ , where  $(Z(\Pi_1(X)))'(f_*)$  is the Reidemeister classes of  $f_*$  restricted to  $Z(\Pi_1(X))$ . This implies that  $J(f) = R(f) = N(f)$ . We have also  $R(f) = R(f_*')$  by Theorem 3.1. Therefore, we have  $R(f_*') = J(f) = N(f) = R(f) = \text{order} \left( \frac{Z(\Pi_1(X))}{(1-f_*')(Z(\Pi_1(X)))} \right)$ .

Now we reexamine an example given in [1 : p.108] in our context.

**EXAMPLE 3.3.** Let  $M_g$  be a connected sum of  $(g+1)$ -copies of an 2-dimensional real projective space  $RP^2$ , where  $g \geq 2$ . Then  $M_g = RP_1^{2\# \dots \#}$

$RP^2_{g+1}$  is a connected, aspherical polyhedron. If  $g$  is even, let  $m=g/2$ , then  $\Pi_1(M_g) = \{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m, \varepsilon | [\alpha_1\beta_1][\alpha_2\beta_2]\dots[\alpha_m\beta_m]\varepsilon^2=1\}$ , where  $[\alpha_i\beta_i]=\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$ . If  $g$  is odd, let  $m=\frac{g-1}{2}$ ,  $\Pi_1(M_g) = \{\alpha_1, \dots, \alpha_{m+1}, \beta_1, \dots, \beta_m, \varepsilon | [\alpha_1\beta_1]\dots[\alpha_m\beta_m]\alpha_{m+1}\varepsilon\alpha_{m+1}^{-1}\varepsilon=1\}$ . For any  $g \geq 2$ , let  $f : M_g \rightarrow M_g$  be a map so that  $f_*(\alpha_1) = \alpha_1^n$ ,  $n > 1$ ,  $f_*(\alpha_j) = 1$  for  $j \neq 1$ ,  $f_*(\beta_k) = 1$  for all  $k$ , and  $f_*(\varepsilon) = 1$ . Then we have  $L(f) \neq 0$ , and the image  $f_*(\Pi_1(M_g)) = \{\alpha_1^n\}$  is an abelian subgroup of  $\Pi_1(M_g)$ . Let  $H$  denote the image group  $\{\alpha_1^n\}$ . Then the centralizer of  $H$  in  $\Pi_1(M_g)$ ,  $Z(\{\alpha_1^n\}, \Pi_1(M_g))$  contains  $H$ . Thus, we have  $H \subset Z(\{\alpha_1^n\}, \Pi_1(M_g)) \subset T(f) \subset \Pi_1(M_g)$ , [1, p.102]. Then by Corollary 2.4, there is a bijective mapping between the Reidemeister classes;  $H'(f_*') \rightarrow (T(f))'(f_{*|T(f)}) \rightarrow (\Pi_1(M_g))'(f_*)$ , when  $f_*' = f_{*|H}$ . Therefore, the cardinality of  $H'(f_*')$  = the cardinality of  $(\Pi_1(M_g))'(f_*) = R(f) = J(f)$ . We also know that cardinality of  $H'(f_*') = |1-n|$ , [1 : p.104]. Hence we have  $|1-n| = J(f) \leq N(f) \leq R(f) = |1-n|$ , that is,  $N(f) = |1-n|$ .

**4, Product theorems of fiber-preserving maps**

Let  $\mathcal{F} = (E, p, B)$  be an orientable Hurewicz regular fibering, where  $E, B$ , and  $p^{-1}(b)$ ,  $b \in B$ , are connected, compact, metric ANR's. A fibering is said to be orientable if the translation map  $\tau : p^{-1}(b) \rightarrow p^{-1}(b)$ , defined by  $\tau(e) = \lambda(e, \omega)(1)$  for every loop  $\omega$  at  $b$  in  $B$ , is homotopic to the identity, where  $\lambda$  is a regular lifting function for the fibering  $\mathcal{F}$ . If a continuous map  $f : E \rightarrow E$  preserves fibers, that is,  $f(p^{-1}(b)) \subset p^{-1}(b')$ , then it induces a continuous map  $\bar{f} : B \rightarrow B$  such that  $\bar{p}f = \bar{f}p$ , and a continuous map  $f_b : p^{-1}(b) \rightarrow p^{-1}(b)$ , for each  $b \in B$ , defined by  $f_b(e) = \lambda(f(e), \omega)(1)$  for each  $e \in p^{-1}(b)$ , where  $\omega$  is a path from  $\bar{f}(b)$  to  $b$  in  $B$ . The triple  $(f, \bar{f}, f_b) : \mathcal{F} \rightarrow \mathcal{F}$  is called a fiber-preserving map triple. It is known that the Nielsen number,  $N(f_b)$ , of the map  $f_b$  is independent of the choice of paths  $\omega$  from  $\bar{f}(b)$  to  $b$  and points  $b$  in  $B$ , [2] and [6]. It is well known that Lefschetz numbers of a fiber-preserving map triple satisfy the relation  $L(f) = L(\bar{f}) \cdot L(f_b)$ ,  $b \in B$ . However, the corresponding result for Nielsen numbers is false, [4], [12]. If the space involved in  $\mathcal{F}$  have abelian fundamental groups, then there is a complete solution to the problem of product relations between Nielsen numbers, [5], [11], [12], and [13]. That is, if  $L(f) \neq 0$ , then there exists an invariant  $P(f)$  such that  $N(f) \cdot P(f) = N(\bar{f}) \cdot N(f_b)$ . If the fundamental groups of the spaces involved are not abelian, then only partial solutions can be found in [2], [4], [5], [7] and [11].

We assume here that the fundamental groups of fibers are abelian, and that the homomorphism  $i_* : \Pi_1(p^{-1}(b)) \rightarrow \Pi_1(E)$ , induced by the inclusion  $i : p^{-1}(b) \rightarrow \Pi_1(E)$ , sends  $\Pi_1(p^{-1}(b))$  into the center of  $\Pi(E)$ . As in section

3,  $\Phi(g)$  denote the set of fixed points of a self map  $g : X \rightarrow X$ , and  $\Phi'(g) = \{F_1, \dots, F_n\}$  denote the set of fixed point classes of  $g$ . We also denote the Reidemeister classes of  $g_\#$  on  $\Pi_1(X)$  by  $(\Pi_1(X))'(g_\#)$ . If  $L(f) \neq 0$ , then there exists at least one essential fixed point classes of  $g$ , and there exists a one-to-one function  $\phi : \Phi'(g) \rightarrow (\Pi_1(X))'(g_\#)$ . We note that  $\phi$  is onto if  $T(g) = \Pi_1(X)$ .

Now we prove a product theorem for Nielsen numbers for a fiber-preserving map triple.

**THEOREM 4.1.** *Let  $\mathcal{F} = (E, p, B)$  be as before, and let  $(f, \bar{f}, f_b)$  be a fiber-preserving map triple such that  $L(f) \neq 0$ . Assume  $\Pi_1(p^{-1}(b))$  is abelian and induced homomorphism  $i_\# : \Pi_1(p^{-1}(b)) \rightarrow \Pi_1(E)$  sends  $\Pi_1(p^{-1}(b))$  into the center of  $\Pi_1(E)$ . Then there exists a constant  $P(f)$  such that  $N(f) \cdot P(f) = N(\bar{f}) \cdot N(f_b)$ .*

*Proof:* Since  $L(f) \neq 0$ , there exists at least one essential fixed point class  $F$  of  $f$ . We know that there exists a one-to-one function  $\phi : \Phi'(f) \rightarrow (\Pi_1(E))'(f_\#)$ . Let  $F \in \Phi'(f)$  be an essential fixed point class containing a point  $e$ . Then  $p(F)$  is also an essential fixed point class of  $\bar{f}$  in  $B$  containing  $p(e) = b$ . This is done in [2] for a locally trivial fibering on polyhedra. But by Fadell [5] and by [11], it is also true for a Hurewicz fibering on compact, connected, ANR spaces. We map the essential fixed point classes of  $f_b$  in  $p^{-1}(b) \cap F (\neq \emptyset)$  into  $F \in \Phi'(f)$ . We do this for every essential fixed point classes in  $\Phi'(f)$  such that  $p^{-1}(b) \cap F \neq \emptyset$ . Then this function induces a function from the set of all essential fixed point classes in  $\Phi'(f_b)$  to the set of all essential fixed point classes in  $\Phi'(f)$ . Since there is a one-to-one map between fixed point classes into the Reidemeister classes, we have

$$\begin{aligned}
 e(\Phi'(f_b)) &\xrightarrow[1:1]{\phi} (\Pi_1(p^{-1}(b)))'(f_{b\#}) \xrightarrow{i^\#} (Z(\Pi_1(E)))'(f_\#) \\
 &\subset (\Pi_1(M))'(f_\#) \xleftarrow[1:1]{\phi} e(\Phi'(f)),
 \end{aligned}$$

where  $e(\Phi'(f))$  denote the set of all essential fixed point classes in  $\Phi'(f)$ , and  $i^\#$  is induced by the homomorphism  $i_\# : \Pi_1(p^{-1}(b)) \rightarrow \Pi_1(E)$ . Since  $\Pi_1(p^{-1}(b))$  and  $Z(\Pi_1(M))$  are abelian,  $i^\#$  is a homomorphism. Therefore, the order of  $i^{\#-1}(c)$  for any  $c \in (Z(\Pi_1(E)))'(f_\#)$  are the same. Let  $P(f)$  denote this number. Thus the number of essential fixed point classes of  $f_b$  in  $p^{-1}(b) \cap F$  which are mapped to  $F$  is exactly  $P(f)$ , and this number is a constant for each essential fixed point class  $F$  in  $\Phi'(f)$  such that  $p^{-1}(b) \cap F \neq \emptyset$ . Then over each essential fixed point class of  $\bar{f}$  in  $(\Pi_1(B))'(\bar{f}_\#)$ ,

there lies precisely  $N(f_b)/P(f)$  essential fixed point classes of  $f$ . Therefore, we have  $N(f) = N(\bar{f}) \cdot N(f_b)/P(f)$ , since  $\bar{f}$  has  $N(\bar{f})$  essential fixed point classes.

**COROLLARY 4.2.** *In the above theorem, if the induced homomorphism  $i_* : \Pi_1(p^{-1}(b)) \rightarrow \Pi_1(E)$  is a monomorphism and  $f_*(Z(\Pi_1(E))) \subset i_*(\Pi_1(p^{-1}(b)))$ , then  $P(f) = 1$ , hence we have  $N(f) = N(\bar{f}) \cdot N(f_b)$ .*

This follows by Corollary 2.4.

An example of this kind is given in [11]. We repeat it here for the readers.

**EXAMPLE 4.3.** Let  $(M, T^h)$  be a toral group action on a compact, connected orientable aspherical manifold. Then the orbit space  $B = M/T^h$  is an aspherical manifold and  $F = (M, p, B)$  becomes a (singular) fibering, where  $p : M \rightarrow B$  is the projection map. Let  $f : M \rightarrow M$  be a fiber preserving map such that  $L(f) \neq 0$ . Then there exists at least one essential fixed point class  $F$  of  $f$ . Let  $e \in F$  and  $p(e) = b \in p(F)$  in  $B$ . Then  $p(F)$  is also an essential fixed point class of  $\bar{f}$  in  $B$ . If  $\alpha : T^h (\cong p^{-1}(b)) \rightarrow M$  is defined by  $\alpha(t) = te$ , for all  $t \in T^h$ , then  $\alpha$  induces a monomorphism  $\alpha_* : \Pi_1(T^h) \rightarrow \Pi_1(M)$ , which sends  $\Pi_1(T^h)$  into the center of  $Z$  of  $\Pi_1(M)$ . Therefore, by Theorem 4.1, there exists a number  $P(f)$  such that  $N(f)P(f) = N(\bar{f}) \cdot N(f_b)$ . In particular, if  $f_*(Z) \subset \alpha_*(\Pi_1(T^h))$ , then we have a product relation,  $N(f) = N(\bar{f}) \cdot N(f_b)$ .

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