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# ROTATIONS AND TRANSLATIONS IN YEH-WIENER SPACE

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## 1. Introduction

Let $R=\{(s,t):\ a\le s\le b,\ \alpha\le t\le\beta\}$ and $C_2[R]$ be Yeh-Wiener space, i.e. $C_2[R]=\{x(\cdot,\cdot):x(a,t)=x(s,\alpha)=0, x(s,t)$ is continuous on $R\}$. $C_2[R]$ is often referred to as two parameter Wiener space. Let $a=s_0<s_1<\cdots<s_m=b$ and $\alpha=t_0<t_1<\ldots<t_n=\beta$ and let $-\infty\le a_{j,k}\le b_{j,k}\le+\infty$ be given for $j=1,2,\ldots,m$ and $k=1,2,\ldots,n$. Let $E=(a_{11},b_{11}]\times\cdots\times(a_{mn},b_{mn}]$. $I=J_{(\vec{s},\vec{t})}(E)\equiv\{x\in C_2[R];\ (x(s_1,t_1),\ldots,x(s_m,t_n)\in E)$ is called a strict interval of $C_2[R]$. If $E$ is an arbitrary measurable subset of $\mathbf{R}^{mn}$ then $I$ is called an interval of $C_2[R]$.

The collection $\mathcal{I}$ of all such strict intervals form a semi-algebra of subtets of $C_2[R]$. The measure of the strict interval $I$ is defined to be

$$m_1(I)=\int_E\omega(\vec{u}:\vec{s}:\vec{t})\,d\vec{u}$$

where

$$\begin{aligned}\omega(\vec{u}:\vec{s}:\vec{t})&=\omega(u_{11},\ldots,u_{mn}:s_1,\ldots,s_m:t_1,\ldots,t_n)\\&=\prod_{j=1}^{m}\prod_{k=1}^{n}\{\pi(s_j-s_{j-1})(t_k-t_{k-1})\}^{-1/2}\\&\quad\cdot\exp\left\{\frac{-(u_{jk}-u_{j-1,k}-u_{j,k-1}+u_{j-1,k-1})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\right\}\end{aligned}$$

and $u_{0,k}=u_{j,0}=u_{0,0}=0$ for all $j$ and $k$. This measure is countably additive on $\mathcal{I}$ and can be extended in the usual way to the $\sigma$-algebra $\sigma(\mathcal{I})$ generated by the strict intervals and then can be further extended so as to be a complete measure. This completed measure space is denoted by $(C_2[R],\mathcal{Y}_1,m_1)$ and $\mathcal{Y}_1$, is called the class of Yeh-Wiener measurable sets.

For $x\in C_2[R]$, let $\|x\|=\max_{(s,t)\in R}|x(s,t)|$. Then $(C_2[R],\|\cdot\|)$ is a separable Banach space.

Let $\mathcal{B}$ be the collection of all sets of the form $J_{(\vec{s},\vec{t})}(B)$ for all $(\vec{s},\vec{t})$ and all Borel set $B$ in $\mathbf{R}^{mn}$. Then $\mathcal{B}$ is an algebra of subsets of $C_2[R]$. Let $\sigma(\mathcal{B})$ be the $\sigma$-algebra generated by $\mathcal{B}$ and $\mathcal{B}(C_2[R])$ be the class of Borel sets in $C_2[R]$. Then it is well known that $\sigma(\mathcal{I})=\sigma(\mathcal{B})=\mathcal{B}(C_2[R])$. $\sigma(\mathcal{I})$ is sometimes referred to as the $\sigma$-algebra of strictly Yeh-Wiener measurable

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sets.

Skoug [8] showed that if $\phi(\sqrt{p^2+q^2}\,x)$ is Yeh-Weiener integrable, then $\phi(px+qy)$ is integrable on $(C_2[R]\times C_2[R],\ \overline{\mathcal{Y}_1\times\mathcal{Y}_1})$ and

$$\int_{C_2[R]\times C_2[R]}\phi(\sqrt{p^2+q^2}\,w)\,dm_1(w) = \int_{C_2[R]\times C_2[R]}\phi(px+qy)\ d(m_1\times m_1)(x,y). \tag{1.1}$$

In section 3 we obtain the converse of this result. In particular we show that $\phi(\sqrt{p^2+q^2}\,\omega)$ is $m_1$-measurable iff $\phi(px+qy)$ is $m_1\times m_1$-measurable and (1.1) holds.

In [8] Skoug used the pathology of scale change transformations in Yeh-Wiener space to show that almost no translations preserve Yeh-Wiener measurability. Specifically, he obtained a set $E$ in $\mathcal{Y}_1$ such that $T_zE\equiv E+z$ is not in $\mathcal{Y}_1$ for $m_1$-a. e. $z$ in $C_2[R]$. In section 4 we obtain several facts which fill in this picture.

## 2. Preliminaries

Let $\sigma_m$ be the partition:

$$\sigma_m=\{(s_j,t_k): s_j=a+\frac{j(b-a)}{m},\quad t_k=\alpha+\frac{k(\beta-\alpha)}{m}: j,k=1,\ 2,\ \dots,\ m\}.$$

For each $x\in C_2[R]$, let

$$S_{\sigma_m}(x)=\sum_{j=1}^{m}\sum_{k=1}^{m}\{x(s_j,t_k)-x(s_{j-1},t_k)-x(s_j,t_{k-1})+x(s_{j-1},\ t_{k-1})\}^2.$$

For each $\lambda\geq 0$, let

$$C_\lambda=\{x\in C_2[R]:\ \lim_{n\to\infty} S_{\sigma_{2^n}}(x)=\lambda^2(b-a)(\beta-\alpha)/2\}$$

$$D=\{x\in C_2[R]:\ \lim_{n\to\infty} S_{\sigma_{2^n}}(x)\ \text{fails to exist}\}.$$

Note that $\nu C_\lambda=C_{\nu\lambda}$ for $\nu>0$, $\lambda\geq 0$, Clearly $C_\lambda(\lambda\geq 0)$ and $D$ are Borel sets and $C_2[R]$ is the disjoint union of this family of sets and $m_1(C_1)=1$ [8].

Let $m_\lambda$ be the Borel measure given by $m_\lambda(B)=m_1(\lambda^{-1}B)$ for $B\in\mathcal{B}(C_2[R])$. Since $\lambda^{-1}C_\lambda=C_1$, $m_\lambda(C_\lambda)=m_1(C_1)=1$.

Let $\mathcal{Y}_\lambda$ denote the $\sigma$-algebra obtained by completing $(C_2[R],\mathcal{B}(C_2[R],m_\lambda)$ and let $\eta_\lambda$ be the class of $m_\lambda$-null sets. Note that every subset of $C_2[R]\backslash C_\lambda$ is in $\eta_\lambda$. Let $\mathcal{Y}=\bigcap_{\lambda>0}\mathcal{Y}_\lambda$ and $\eta=\bigcap_{\lambda>0}\eta_\lambda$. Each $K\in\mathcal{Y}$ is called a scale-invariant mesurable set and each $N\in\eta$ is called a scale-invariant null set [4].

The following four propositions are well known results. We will state them without proof [4, 10].

PROPOSITION 2.1. *$E$ is Lebesgue measurable in $\mathbf{R}^{nm}$ iff $J_{(\vec{s},\vec{t})}(E)$ is Yeh-Wiener measurable. In this case,*

$$m_1(J_{(\vec{s},\vec{t})}(E))=\int_E\omega(\vec{u}:\vec{s}:\vec{t})\,d\vec{u}.$$

PROPOSITION 2. 2. *Let $f(u_{11}, \ldots, u_{mn})$ be a Lebesgue measurable funtion on $\mathbf{R}^{mn}$ and $F(x)=f(x(s_1, t_1), \ldots, x(s_m, t_n))$. Then $F$ is Yeh-Wiener measurable and*

$$\int_{C_2[R]} F(x)\,dm_1(x) \overset{*}{=} \int_{R^{mn}} f(u)\,\omega(u:s:t)\,du.$$

REMARK. Throughout this paper, by $\overset{*}{=}$ we mean that if either side exists then both sides exist and they are equal.

PROPOSITION 2. 3. (a) *If $E$ is Yeh-Wiener measurable, then $-E$ is Yeh-Wiener measurable and $m_1E=m_1(-E)$.*

$$\text{(b)} \int_{C_2[R]} F(x)\,dm_1(x)=\int_{C_2[R]} F(-x)\,dm_1(x).$$

PROPOSITION 2. 4. (i) *$N$ is in $\eta_\lambda$ iff $\lambda^{-1} N$ is in $\eta_1$; equivalently, $\eta_\lambda=\lambda\eta_1$.*
(ii) *$E$ is in $\mathcal{Y}_\lambda$ iff $\lambda^{-1} E$ is in $\mathcal{Y}_1$; equivalently, $\mathcal{Y}_\lambda=\lambda\mathcal{Y}_1$.*
(iii) *$m_\lambda(E)=m_1(\lambda^{-1}E)$ for $E$ in $\mathcal{Y}_\lambda$.*

## 3. Rotations in the product of two Yeh-Wiener spaces.

Let $R_\theta$ denote the linear transformation from $\mathbf{R}^2$ onto $\mathbf{R}^2$ which rotates each vector through an angle $\theta$. Define $R_\theta^*: C_2[R]\times C_2[R]\to C_2[R]\times C_2[R]$ by $(X, Y)=R_\theta^*(x, y)$ where $(X(s,t), Y(s,t))=R_\theta(x(s,t), y(s,t))$ for all $(s,t)\in R$. Then

$$X(s,t)=x(s,t)\cos\theta-y(s,t)\sin\theta,$$
$$Y(s,t)=x(s,t)\sin\theta+y(s,t)\cos\theta.$$

Then $R^*$ is 1-1, onto, linear and continuous. It is an isometry and hence it carries the Borel class onto the Borel class.

The following lemma which is useful in the proof of Proposition 3. 2 is taken from [1, Theorem 3. 1. 1, p. 109].

LEMMA 3. 1. *For each $i=1, 2, \ldots, n$, let $(\Omega_i, \mathcal{U}_i)$ be a measurable space and let $\mathcal{A}_i$ be a generator of the $\sigma$-algebra $\mathcal{U}_i$ which contains a sequence $\{E_{ik}\}_{k=1}^{\infty}$ of sets with $E_{ik}\uparrow\Omega_i$. Then the $\sigma$-algebra $\mathcal{U}_1\times\mathcal{U}_2\times\ldots\times\mathcal{U}_n$ is generated by the system of all sets $E_1\times E_2\times\ldots\times E_n$, with $E_i\in\mathcal{A}_i$ $(i=1, 2, \ldots, n)$.*

PROPOSITION 3. 2. *$m_1\times m_1=(m_1\times m_1)\circ R^{*}{}_\theta^{-1}$ on $\mathcal{B}(C_2[R]\times C_2[R])$.*

*Proof.* It follows from Lemma 3. 1, Proposition 2. 2 and the fact that $\mathcal{B}(C_2[R]\times C_2[R])=\mathcal{B}(C_2[R])\times\mathcal{B}(C_2[R])$.

COROLLARY 3. 3. (a) *If $N$ is a null set in $\mathcal{B}(C_2[R]\times C_2[R])$, then $R^{*}{}_\theta^{-1}(N)$ is a null set in $\mathcal{B}(C_2[R]\times C_2[R])$.*
(b) *$R^*{}_\theta$ is $m_1\times m_1$-measurable.*
(c) *$m_1\times m_1=(m_1\times m_1)\circ R^{*}{}_\theta^{-1}$ on $\overline{\mathcal{Y}_1\times\mathcal{Y}_1}$.*

The following results follow from Corollary 3.3, the change of variables theorem [5, Theorem C., p. 163], and the Fubini theorem.

THEOREM 3.4. *Let $F$ be a real valued function on $C_2[R]\times C_2[R]$. Then $F(X, Y)$ is $m_1\times m_1-$measurable iff $F(R^*_\theta(x, y))$ is $m_1\times m_1-$measurable and in this case, we get*

$$\int_{C_2[R]\times C_2[R]} F(X, Y)\,d(m_1\times m_1)(X, Y) \overset{*}{=} \int_{C_2[R]\times C_2[R]} F(R^*_\theta(x, y))\,d(m_1\times m_1)(x, y).$$

*Proof.* Suppose $F$ is $m_1\times m_1-$measurable. Then $F^{-1}(B)$ is in $\overline{\mathcal{U}_1\times\mathcal{U}_1}$ for each Borel set $B$ in $\mathbf{R}$. By Corollary 3.3, $R^{*}_\theta{}^{-1}F^{-1}(B)$. is in $\overline{\mathcal{U}_1\times\mathcal{U}_1}$. Hence $F\circ R^*_\theta$ is $m_1\times m_1-$measurable. Conversely, let $F\circ R^*_\theta$ be $m_1\times m_1-$measurable. Then $R^{*}_\theta{}^{-1}\circ F^{-1}(B)$ is in $\overline{\mathcal{U}_1\times\mathcal{U}_1}$ for each Borel set $B$ in $\mathbf{R}$. By Corollary 3.3, $F^{-1}(B)$ is in $\overline{\mathcal{U}_1\times\mathcal{U}_1}$ and hence $F$ is $m_1\times m_1-$measurable.

$$\begin{aligned}&\int_{C_2[R]\times C_2[R]} F(R^*_\theta(x, y))\,d(m_1\times m_1)(x, y)\\ &=\int_{C_2[R]\times C_2[R]} F(X, Y)\,d(m_1\times m_1)R^{*}_\theta{}^{-1}(X, Y)\\ &=\int_{C_2[R]\times C_2[R]} F(X, Y)\,d(m_1\times m_1)(X, Y).\end{aligned}$$

The first equality is an immediate consequence of the change of variable theorem [5, Theorem C, p. 163] and the second equality holds by Corollary 3.3.

REMARK. J. Bearman [2] showed that if $F(X, Y)$ is an integrable functional on the product of two Wiener spaces, then the integration formula holds in the product of two Wiener spaces. C. Park [7] extended Bearman's theorem to the product of two $n$-dimensional Yeh-Wiener spaces. Theorem 3.4 shows that the converse of this result is also true.

THEOREM 3.5. *Let $\phi$ be a real valued function on $C_2[R]$. $\phi$ is $m_1-$measurable iff $\phi(x\sin\theta + y\cos\theta)$ is $m_1\times m_1-$measurable and in this case*

$$\int_{C_2[R]}\phi(Y)\,dm_1(Y) =\int_{C_2[R]\times C_2[R]} \phi(x\sin\theta+y\cos\theta)\,d(m_1\times m_1)(x, y).$$

*Proof.* Let $\pi$ be a projection: $C_2[R]\times C_2[R]\to C_2[R]$ defined by $\pi(x, y) =y$. Then $\phi(x\sin\theta + y\cos\theta) =\phi\circ\pi\circ R^*_\theta(x, y)$. For any Borel set $B$ in $\mathbf{R}$, $\pi^{-1}(\phi^{-1}B) =C_2[R]\times\phi^{-1}B$ and $(\phi\circ\pi\circ R^*_\theta)^{-1}(B) =R^{*}_\theta{}^{-1}(C_2[R]\times\phi^{-1}B)$. Suppose $\phi$ is $m_1-$measurable. Then $\phi^{-1}B$ is in $\mathcal{U}_1$ and $(\phi\circ\pi\circ R^*_\theta)^{-1}(B)$ is in $\overline{\mathcal{U}_1\times\mathcal{U}_1}$.

Hence $\phi(x\sin\theta+y\cos\theta)$ is $m_1\times m_1$ measurable. Conversely, if $\phi(x\sin\theta+y\cos\theta)$ is $m_1\times m_1$-measurable, then $R^{*}{}_{\theta}^{-1}(C_2[R]\times C_2[R])=(\phi\circ\pi\circ R^{*})^{-1}(B)$ is in $\overline{\mathcal{Y}_1\times\mathcal{Y}_1}$ for each Borel set $B$ in $\mathbf{R}$. By Corollary 3.3, $C_2[R]\times\phi^{-1}B$ is in $\overline{\mathcal{Y}_1\times\mathcal{Y}_1}$. Note that $x$-cross section of $C_2[R]\times\phi^{-1}B=\phi^{-1}B$ for all $x$ in $C_2[R]$. Hence by Fubini theorem, for almost all $x$, $\phi^{-1}B$ is $m_1$-measurable and so $\phi^{-1}B$ is in $\mathcal{Y}_1$. Let $F(X,Y)=\phi(Y)$. Then

$$\int_{C_2[R]}\phi(Y)dm_1(Y)=\int_{C_2[R]\times C_2[R]}F(X,Y)d(m_1\times m_1)(X,Y)$$
$$=\int_{C_2[R]\times C_2[R]}F(x\cos\theta-y\sin\theta,\ x\sin\theta+y\cos\theta)d(m_1\times m_1)(x,y)$$
$$=\int_{C_2[R]\times C_2[R]}\phi(x\sin\theta+y\cos\theta)d(m_1\times m_1)(x,y).$$

THEOREM 3.6. *Let $p$ and $q$ be real numbers. Then $\phi(\sqrt{p^2+q^2}\,\omega)$ is $m_1$-measurable iff $\phi(px+qy)$ is $m_1\times m_1$-measurable and in this case,*

$$\int_{C_2[R]}\phi(\sqrt{p^2+q^2}\omega)dm_1(\omega)$$
$$=\int_{C_2[R]\times C_2[R]}\phi(px+qy)d(m_1\times m_1)(x,y).$$

*Proof.* Choose $\theta$ so that $\sin\theta=p/\sqrt{p^2+q^2}$ and $\cos\theta=q/\sqrt{p^2+q^2}$. Let $\psi$: $C_2[R]\to C_2[R]$ be defined by $\psi(w)=\sqrt{p^2+q^2}w$. Apply Tneorem 3.5 to the function $\phi\circ\psi$. Then $\phi\circ\psi(w)=\phi(\sqrt{p^2+q^2}\,w)$ is $m_1$-measurable iff $\phi(px+qy)$ $=\phi\circ\psi(x\sin\theta+y\cos\theta)$ is $m_1\times m_1$-measurable.

$$\int_{C_2[R]}\phi(\sqrt{p^2+q^2}\,w)dm_1(w)$$
$$=\int_{C_2[R]\times C_2[R]}\phi(\sqrt{p^2+q^2}\ (x\sin\theta+y\cos\theta))\ d(m_1\times m_1)(x,y)$$
$$=\int_{C_2[R]\times C_2[R]}\phi(px+qy)\ d(m_1\times m_1)(x,y).$$

REMARK. Skoug [8] showed that if $\phi(\sqrt{p^2+q^2}\,x)$ is Yeh-Wiener integrable, then $\phi(px+qy)$ is integrable on $(C_2[R]\times C_2[R],\ \overline{\mathcal{Y}_1\times\mathcal{Y}_1})$ and the integral formula holds. Theorem 3.6 shows that the converse of this result is also true.

## 4. Translations in Yeh-Wiener space

In [8] Skoug used the pathology of scale change transformations in Yeh-Wiener space to show that almost no translations preserve Yeh-Wiener measurability. Specifically, he obtained a set $E$ in $\mathcal{Y}_1$ such that $T_zE\equiv E+z$ is not in $\mathcal{Y}_1$ for $m_1$-a.e. $z$ in $C_2[R]$. We obtain several facts below which fill in this picture. For example, we will see that if $E$ is in $\mathscr{U}_{\sqrt{2}}$ then $E+z$ is in $\mathscr{U}_p$ for $m_q$-a. e. $z$. More generally, it will be seen that if $E$ is in $\mathscr{U}_{\sqrt{p^2+q^2}}$, then $E+z$ is in $\mathscr{U}_p$ for $m_q$-a. e. $z$. First we need the following result which follows from Proposition 2. 4, the change of variables theorm, and Theorem 3. 6.

THEOREM 4. 1. *Let $p$ and $q$ be positive numbers, The following assertions are equivalent*:

(a) *$f(\sqrt{p^2+q^2}\ z)$ is an $m_1$-measurable function of $z$.*

(b) *$f(z)$ is an $m_{\sqrt{p^2+q^2}}$-measurable function of $z$.*

(c) *$f(x+y)$ is an $m_p\times m_q$-measurable function of $x$ and $y$.*

(d) *$f(px+qy)$ is an $m_1\times m_1$-measusable function of $x$ and $y$.*

*If any one (and hence all) of* (a)$\sim$(d) *holds, then*

$$\int_{C_2[R]} f(\sqrt{p^2+q^2}\ z)\,dm_1(z) \overset{*}{=} \int_{C_2[R]} f(z)\,dm_{\sqrt{p^2+q^2}}(z)$$
$$\overset{*}{=} \int_{C_2[R]\times C_2[R]} f(x+y)\,d(m_p\times m_q)(x,y)$$
$$\overset{*}{=} \int_{C_2[R]\times C_2[R]} f(px+qy)\,d(m_1\times m_1)(x,y).$$

*Proof.* (1) The equivalence of (a) and (b). Define $T\colon (C_2[R], \mathscr{U}_1, m_1)\to (C_2[R], \mathscr{U}_{\sqrt{p^2+q^2}}.\ m_{\sqrt{p^2+q^2}})$ by $T(z)=\sqrt{p^2+q^2}\,z$. Then $T$ is a measurable transformation, for let $E\in\mathscr{U}_{\sqrt{p^2+q^2}}$, then $T^{-1}E=(1/\sqrt{p^2+q^2})E$ is in $\mathscr{U}_1$ by Proposition 2. 4. For any real number $\alpha$, $(f\circ T)^{-1}(\alpha,\infty)$ is in $\mathscr{U}_1$ iff $f^{-1}(\alpha,\infty)$ is in $\mathscr{U}_{\sqrt{p^2+q^2}}$ by Proposition 2. 4. Hence (a) and (b) are equivalent and

$$\int_{C_2[R]} f(z)\,dm_{\sqrt{p^2+q^2}}(z) = \int_{C_2[R]} f(z)\,dm_1\circ T^{-1}(z)$$
$$= \int_{C_2[R]} f\circ T(z)\ dm_1(z) = \int_{C_2[R]} f(\sqrt{p^2+q^2}\,z)\ dm_1(z).$$

The first and second equalities follow from Proposition 2. 4 and the change of variable theorem [5], respectively.

(2) Similarly one can show the equivalence of (c) and (d) and the corresponding integral equation. The equivalence of (a) and (d) and its integral equation are the consequences of Theorem 3. 6.

The next result follows immediately from Theorem 4. 1, Proposition 2. 3. and the Fubini Theorem, if we take $f$ to be the characteristic function of $A$ where $A$ is in $\mathscr{U}_{\sqrt{p^2+q^2}}$.

THEOREM 4. 2. *Let $p$ and $q$ be positive numbers and let $A$ be in $\mathscr{U}_{\sqrt{p^2+q^2}}$. Then $A+y$ and $A-y$ are in $\mathscr{U}_p$ for $m_q$-a. e. $y$ and $m_p(A+y)$ and $m_p(A-y)$ are $m_q$-measurable of $y$. Similarly $A+x$ and $A-x$ are in $\mathscr{U}_q$ for $m_p$-a. e. $x$ and $m_q(A+x)$ and $m_q(A-x)$ are $m_p$-measurable functions of $x$. Furthermore,*

$$\int_{C_2[R]} m_p(A+y)\,dm_q(y) = \int_{C_2[R]} m_p(A-y)\,dm_q(y)$$
$$= (m_p \times m_q)(\{(x,y) : x+y \in A\})$$
$$= m_{\sqrt{p^2+q^2}}\ (A)$$
$$= \int_{C_2[R]} m_q(A-x)\,dm_p(x)$$
$$= \int_{C_2[R]} m_q(A+x)\,dm_p(x).$$

COROLLARY 4. 3. *Let $A$ be in $\mathcal{Y}_{\sqrt{2}}$. Then $A+y$ is in $\mathcal{Y}_1$ for $m_1$*-a. e. $y$.

COROLLARY 4. 4. *Let $p$ and $q$ be positive numbers. The translasion map $T_y : (C_2[R], \mathcal{Y}_q, m_q) \to (C_2[R], \mathcal{Y}_p, m_p)$ defined by $T_y(x) = x+y$ is $m_q$-almost never measurability preserving.*

*Proof.* Applying Theorem 4. 2 with $A = C_{\sqrt{p^2+q^2}}$, we see that $y + C_{\sqrt{p^2+q^2}}$ is in $\mathcal{Y}_p$ and

$$\int_{C_2[R]} m_p(y + C_{\sqrt{p^2+q^2}})\,dm_q(y) = 1.$$

Hence $m_p(y+C_{\sqrt{p^2+q^2}}) = 1$ for $m_q$-a. e. $y$. *Let $M$* be a $m_p$-non measurable subset of $C_p$. For each $y$ such that $m_p(y+C_{\sqrt{p^2+q^2}}) = 1$, let $M_y = M \cap (y + C_{\sqrt{p^2-q^2}})$. Then $M_y$ is $m_p$-non measurable. Let $G_y = M_y - y$. Since $m_q(C_{\sqrt{p^2+q^2}}) = 0$ and $G_y \subset C_{\sqrt{p^2+q^2}}$, $G_y$ is in $\eta_q$ and hence it is in $\mathcal{Y}_q$. But $y + G_y = M_y$ is not in $\mathcal{Y}_p$.

COROLLARY 4. 5. $(m_p \times m_q)(\{x, y) \in C_2[R] \times C_2[R] : x+y \in C_{\sqrt{p^2+q^2}}\}) = 1.$
*In particular, $x+y$ is in $C_{\sqrt{2}}$ for $m_1 \times m_1$*-a. e. $(x, y)$.
*In contrast,*

$$(m_p \times m_q)(\{x, y) : x+y \in C_\lambda \neq \sqrt{p^2+q^2}\}) = 0.$$

Next we give some positive results concerning the translation of scale-invariant measurable sets and scale-invariant null sets.

COROLLARY 4. 6. *Let $A$ be in $\mathcal{Y}$. Then for each $p > 0$, $A+y$ isin $\mathcal{Y}_p$ with the exception of at most a scale-invariant null set of $y$*'s.

*Proof.* Let $p > 0$ begiven. It sufficies to show that for each $\lambda > 0$, $A+y$ is in $\mathcal{Y}_p$ for $m_\lambda$-a. e. $y$. But $A$ is in $\mathcal{Y}$ implies that $A$ is in $\mathcal{Y}_{\sqrt{p^2+\lambda^2}}$ and so the result follows from Theorem 4. 2.

CROOLLARY 4. 7. *Let $N$ be in $\eta$. Then for each $p > 0$, $N+y$ is in $\eta_p$ with the exception of at most a scale-invariant null set of $y$'s.*

*Proof.* Let $p$ be a given positive real number. It sufficies to show that for each $\lambda > 0$, $m_p(N+y) = 0$ for $m_\lambda$-a. e. $y$. But $N$ in $\eta$ implies that $N$ is in $\mathcal{Y}_{\sqrt{p^2+\lambda^2}}$ and

$$\int_{C_2[R]} m_p(N+y)\,dm_\lambda(y) = m_{\sqrt{p^2+\lambda^2}}\ (N) = 0.$$

Hence $m_p(N+y)=0$ for $m_\lambda$-a. e. $y$.

## References


1. N. Bauer, *Probability Theory and Elements of Measure Theory*, Holt, Rinehart and Winston, New York, 1972.
2. J. E. Bearman, *Rotationsin the product of two Wiener spaces*, Proc. Amer. Math. Soc., **7**(1956), 522–527.
3. R. H. Cameron, *The translation pathology of Wiener space*, Duke Math. J., **21** (1954), 623–628.
4. K. S. Chang, *Scale-invariant measurability in Yeh-Wiener space*, To appear.
5. P. Halmos, *Measure Theory*, Van Nostrand, New York, 1950.
6. G. W. Johnson and D. L. Skoug, *Scale-invariant measurabilitx in* Wiener space, Pacific J. Math., **83**(1979), 157–176.
7. C. Park, *A generalized Paley-Wiener-Zygmund integral and its applications*, Proc. Amer. Math. Soc., **23**(1969), 388–400.
8. D. L. Skoug, *The change of scale and translation pathology in Yeh-Wiener space*, Riv. Mat. Univ. Parma, **3**(1977) 79–87.
9. J. Yeh, *Stochastic Processes and the Wiener Integral*, Marcel Dekker, New York, 1973.
10. _______, *Wiener measure in a space of functions of functions of two variables*, Trans. Amer. Math. Soc., **95**(1960), 433–450.



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