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A POWER FACTORIZATION THEOREM IN BANACH MODULES

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1. Introduction

Since Cohen proved a factorization theorem in a Banach algebra [3], various extensions of this theorem have been appeared [2], [1] and [4]. One way of these extensions is power factorization of the form $x=a^n y_n$. Allan and Sinclair proved in [1] that every element x of the closed linear span of AX, where X is a left Banach A-module of a Banach algebra A with bounded approximate identity for A, can be factored in the form of $x=a^n y_n$, $a\in A$, $y_n \in X$, with some growth condition on the norm of y_n. Then Grønbeck obtained a power factorization theorem similar to Allan and Sinclair's result for a Banach A-module X when A is a commutative radical Banach algebra with a bounded left approximate identity for X. The author understands that it is not known yet whether or not Grønbeck theorem holds without the radicality of Banach algebras.

In this paper we prove a power factorization theorem for a Banach left A-module X when the Banach algebra A has a bounded left approximate identity $\{e_\lambda : \lambda\in \Lambda\}$ for X satisfying the condition $e_\lambda e_\mu = e_\lambda$ whenever $\lambda \leq \mu$. In [2] the radicality of Banach algebra was used to insure the invertibility of certain elements in the algebra. In our theorem the condition imposed on the approximate identity is used to secure the invertibility of the elements of the form $(1-\gamma)^k + \sum_{i=1}^{k} \gamma(1-\gamma)^{i-1} e_i$, $k\in \mathbf{N}$

2. Approximate identities

Let A be a normed algebra over $\mathbf{R}$ or $\mathbf{C}$ and let X be a left A-module. If X is a Banach space and the module multiplication

$$(a, x) \to ax, \quad a\in A, \quad x\in X$$

satisfies the condition: There exists a constant $K>0$ such that

$$\|ax\| \leq K\|a\|\|x\|, \quad a\in A, \quad x\in X$$

then X is called a *Banach left A-module.*

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Let X be an arbitrary Banach space and $A=BL(X)$, the Banach algebra of bounded linear operators on X with operator norm, then X is a Banach left A-module with the module multiplication ax, the evaluation of a at x.

DEFINITION Let X be a Banach left A-module where A is a normed algebra. A net $\{e_\lambda : \lambda\in\Lambda\}$ in A is said to be a *bounded left approximate identity for* X if the net is bounded and $\lim_\lambda\|e_\lambda x-x\|=0$ for all $x\in X$.

EXAMPLE Let S be an infinite set, let $X=l_p(S)$ with $1\leq p<\infty$ and $A=BL(X)$. For a subset F of S let e_F be the projection of $l_p(S)$ onto the closed subspace $l_p(F)$. Then the net

$$P=\{e_F : F \text{ is a finite subset of } S\}$$

with the ordering

$$e_F\leq e_E \text{ if and only if } F\subset E$$

is a bounded left approximate identity for $X=l_p(S)$.

3. A power factorization theorem

In this section A will denote a Banach algebra, $A^\#$ the unitization of A and X a Banach left A-module.

In the above example the approximate identity P satisfies the conditions $e_\lambda e_\mu=e_\lambda=e_\mu e_\lambda$ whenever $\lambda\leq\mu$ and $e_\lambda e_\mu=0$ if and only if $e_\lambda(X)\cap e_\mu(X)=\phi$. In general if these conditions are weakened so that a left approximate idendity for a Banach A-module X satisfies only $e_\lambda e_\mu=e_\lambda$ whenever $\lambda\leq\mu$, we can not show that it is also a left approximate identity for A. However, if A has a bounded left approximate identity for X which satisfies the above weakened condition, then we can prove a power factorization Theorem for X.

LEMMA 1. *Let $\{e_\lambda : \lambda\in\Lambda\}$ be a net in A bounded by $d>0$ satisfying $e_\lambda e_\mu=e_\lambda$ whenever $\lambda\leq\mu$. If γ is a real number with $0<\gamma<(d+1)^{-1}$, then any subsequence $\{e_n : n\in\mathbf{N}\}$ of the net satisfies the following*:

(i) $1-\gamma+\gamma e_n$ *is invertible in* $A^\#$ *for all* $n=1,2,\ldots$, *and we have*

$$\|(1-\gamma+\gamma e_n)^{-1}\|\leq\frac{1}{1-\gamma-\gamma d}$$

for all $n=1,2,\ldots$.

(ii) $b_n=(1-\gamma)^n+\sum_{k=1}^{n}\gamma(1-\gamma)^{k-1}e_k$ *is invertible in* $A^\#$ *for all* $n=1,2,\ldots$.

Proof. (i) Since $0<\gamma<(d+1)^{-1}$ we have $0<d<\frac{1-\gamma}{\gamma}$. Hence $0<\frac{\gamma d}{1-\gamma}<1$. Therefore $\|\frac{\gamma}{1-\gamma}e_n\|<1$ and it follows that

$$1-\gamma+\gamma e_n=(1-\gamma)\left(1+\frac{\gamma}{1-\gamma}e_n\right)$$

is invertible for all $n=1, 2, \dots$. Now, we have

$$(1-\gamma+\gamma e_n)^{-1}=(1-\gamma)^{-1}\sum_{k=0}^{\infty}\left(\frac{\gamma}{1-\gamma}e_n\right)^k$$

Therefore

$$\|(1-\gamma+\gamma e_n)^{-1}\| = \frac{1}{1-\gamma}\sum_{k=1}^{\infty}\left(\frac{\gamma}{1-\gamma}\right)^k\|e_n\|^k \leq \frac{1}{1-\gamma-\gamma d}.$$

(ii) $b_1=1-\gamma+\gamma e_1$ is invertible by (i). Now suppose that

$$b_n=(1-\gamma)^n+\sum_{k=1}^{n}\gamma(1-\gamma)^{k-1}e_k$$

is invertible. We want to show that b_{n+1} is invertible. To do this let

$$f(e)=(1-\gamma+\gamma e)^{-1},\quad E_n(e)=(1-\gamma)^n+\sum_{k=1}^{n}(1-\gamma)^{k-1}e_k f(e).$$

An easy computation shows that

(1) $$(E_n(e_{n+1}))f(e_{n+1})^{-1}=b_{n+1}.$$

On the other hand

$$\begin{aligned}E_n(e_{n+1})-b_n&=\sum_{k=1}^{n}\gamma(1-\gamma)^{k-1}e_k f(e_{n+1})-\sum_{k=1}^{n}\gamma(1-\gamma)^{k-1}e_k\\&=\left(\sum_{k=1}^{n}\gamma(1-\gamma)^{k-1}e_k\right)(1-e_{n+1})\gamma f(e_{n+1})=0\end{aligned}$$

since $e_k e_{n+1}=e_k$ for all $k\leq n+1$. Therefore, we have

$$E_n(e_{n+1})=b_n \text{ for all } n=1, 2, \dots.$$

By this equality we see that $E_n(e_{n+1})$ is invertible and by (1) b_{n+1} is invertible.

In the remainder of this section we assume that A has a bounded left approximate identity $\{e_\lambda : \lambda\in\Lambda\}$ for X such that for some $d>0$ $\|e_\lambda\|\leq d$ for all $\gamma\in\Lambda$ and $e_\lambda e_\mu=e_\lambda$ whenever $\lambda\leq\mu$.

LEMMA 2. *Let $x\in X$. Given $\delta>0$ and a monotone increasing sequence $\{K_n\}$ of positive integers there exists a subsequence $\{e_n : n\in\mathbf{N}\}$ of $\{e_\lambda : \lambda\in\Lambda\}$ such that*

(i) $\lim \|e_n x-x\|=0$,

(ii) $\|b_n^{-j}x-b_{n-1}^{-j}x\|<\delta 2^{-n}$ *for* $j=1, 2, \dots, K_n$,

where $b_0=1$, $b_n=(1-\gamma)^n+\sum_{k=1}^{n}\gamma(1-\gamma)^{k-1}e_k$ *and* $0<\gamma<(d+1)^{-1}$.

Proof. For simplicity let

$$f(e)=(1-\gamma+\gamma e)^{-1} \text{ and } \varepsilon=(1-\gamma-\gamma d)^{-1}$$

for $e\in A$ with $\|e\|\leq d$. By Lemma 1 we have $\|f(e)\|\leq\varepsilon$. For $j=1, 2, \dots,$ K_1

$$(1)\qquad \|f(e)^j x - x\| = \|\sum_{k=1}^{j}\{f(e)^k x - f(e)^{k-1}x\}\| \le \|\sum_{k=1}^{j} f(e)^k\| \|x-(1-\gamma+\gamma e)x\| \le \gamma\sum_{k=1}^{j}\varepsilon^k\|ex-x\|.$$

Now, choose e_1 such that

$$\|e_1x-x\|<\min\left\{1,\ \delta(\gamma\sum_{k=1}^{K1}\varepsilon^k)^{-1}\right\}.$$

Then by (1)

$$\|b_1{}^{-j}x-x\|<\delta,\ j=1,2,\dots,K_1.$$

Suppose that $e_1, e_2, \dots, e_n$ have been inductively chosen so that (ii) holds for each e_k, $k=1,2,\dots,n$. Choose e_{n+1} such that

$$(2)\qquad \|e_{n+1}x-x\|\le\min\left\{\frac{1}{n},\ \frac{\delta}{2^n}(\gamma\|b_n{}^{-1}\|j\sum_{k=1}^{Kn+1}\varepsilon^k)^{-1}\right\}.$$

Since we have

$$(E_n(e_{n+1}))(1-\gamma+\gamma e_{n+1})=b_{n+1},$$
$$E_n(e_{n+1})=b_n,$$

we have

$$b_{n+1}{}^{-j}=b_n{}^{-j}f(e_{n+1})^j,$$

where $f(e)=(1-\gamma+\gamma e)^{-1}$. Therefore, by (1) and (2)

$$\|b_{n+1}{}^{-j}x-b_n{}^{-j}x\|\le\|b_n{}^{-1}\|^j\|f(e_{n+1})^jx-x\| \le\gamma\|b_n{}^{-1}\|^j\sum_{k=1}^{j}\varepsilon^k\|e_{n+1}x-x\| <\delta 2^{-n}$$

for all $j=1,2,\dots,K_{n+1}$. Also by the choice of e_n we have

$$\lim_{n\to\infty}\|e_nx-x\|=0.$$

THEOREM *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers tending to ∞ such that $\alpha_n>1$ for all $n\in\mathbf{N}$ let $K\in\mathbf{N}$ and $\delta>0$. Then for each $x\in X$ there exist $a\in A$, a sequence $\{y_j\}$ in X and a sequence $\{I_k\}$ of intervals with $I_k=[s_k,t_k]$, $s_k\le s_{k+1}$, $t_k\le t_{k+1}$ and $t_k-s_k>\beta_k$ for all $k\in\mathbf{N}$ such that*

(i) $x=a^jy_j$ *for all* $j\in\mathbf{N}$,

(ii) $\|a\|\le d$,

(iii) $\|x-y_j\|\le\delta$ *for all* $j=1,2,\dots,K$,

(iv) $\|y_j\|\le\alpha_j{}^j\|x\|$ *for all* $j\in\bigcup_{k=1}^{\infty}I_k$.

Proof. We may assume that

$$\|ay\| \leq \|a\|\|y\| \text{ for all } a \in A, \ y \in X,$$

and

$$\delta < \min\{\|x\|, \ \alpha_1 - 1, \ \alpha_2 - 1, \ldots\}.$$

We want to choose a subsequence $\{e_n : n \in \mathbf{N}\}$ of the net $\{e_\lambda : \lambda \in \Lambda\}$ and a sequence $\{I_k : k \in \mathbf{N}\}$ of intervals such that

(a) $I_k = [s_k, t_k]$, $t_k - s_k \geq \beta_k$ and $s_k \leq s_{k+1}$, $t_k \leq t_{k+1}$ for all $j \in \mathbf{N}$,

(b) $\|b_n^{-j}x - b_{n-1}^{-j}x\| < \delta 2^{-n}$ for all $n \in \mathbf{N}$, $j \leq t_{n-1}$,

(c) $\|b_n^{-1}\| + 1 \leq \alpha_j$ for all $j \in I_n$ and $n \in \mathbf{N}$.

Here $t_0 = K$, $b_0 = 1$ and $b_n = (1-\gamma)^n + \sum_{k=1}^{n} \gamma(1-\gamma)^{k-1} e_k$.

Let $I_0 = [0, K]$. By Lemma 2 choose e_1 so that

$$\|b_1^{-j}x - b_0^{-j}x\| < \frac{\delta}{2}, \ j = 1, 2, \ldots, K.$$

Since $\alpha_n \to \infty$ we can choose I_1 so that (a) and (c) are satisfied. Suppose that $e_1, e_2, \ldots, e_n$ and intervals $I_1, I_2, \ldots, I_n$ have been chosen to satisfy (a), (b) and (c). Choose e_{n+1} such that

$$\|b_{n+1}^{-j}x - b_n^{-j}x\| < \frac{\delta}{2^n + 1} \text{ for all } j = 1, \ 2, \ldots, \ t_n$$

by Lemma 2, then choose I_{n+1} so that (a) and (c) hold. Let $a = \lim_{n\to\infty} b_n = \sum_{k=1}^{\infty} \gamma(1-\gamma)^{k-1} e_k \in A$. Clearly $\|a\| \leq d$. By (b) $\{b_n^{-j}x\}$ is a Cauchy sequence in X, and

$$y_j = \lim_{n\to\infty} b_n^{-j}x \in X$$

for all $j \in \mathbf{N}$. Hence, we have

$$x = \lim_{n\to\infty} b_n^j (b_n^{-j}x) = a^j y_j$$

for all $j \in \mathbf{N}$. If $1 \leq j \leq K$, then

$$\begin{aligned} \|x - y_j\| &= \|\lim_{n\to\infty} \sum_{k=1}^{n} (b_k^{-j}x - b_{k-1}^{-j}x)\| \\ &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \delta 2^{-n} = \delta. \end{aligned}$$

If $j \in \bigcup_{k=1}^{\infty} I_k$, then $j \in I_n$ for some n and

$$\begin{aligned} \|y_j\| &\leq \|b_n^{-j}x\| + \sum_{i=n}^{\infty} \|b_{i+1}^{-j}x - b_i^{-j}x\| \\ &\leq \|b_n^{-j}x\| + \delta \leq \|x\|(\|b_n^{-1}\|^j + 1) \\ &\leq \|x\|(\|b_n^{-1}\| + 1)^j \\ &\leq \|x\|\alpha_j^j. \end{aligned}$$

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