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NOTE ON MIZOHATA TYPE OPERATORS

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Introduction

In this note we shall generalize a little further Sjöstrand's results on Mizohata type operators developed in [4]. In [4], Sjöstrand considered a germ of the Mizohata type vector field

$$L=\frac{\partial}{\partial t}-itg(t,x)\frac{\partial}{\partial x},\quad \operatorname{Re} g(0,0)\neq 0 \tag{$*$}$$

defined near the origin in $\mathbf{R}^2$, where g has even Taylor expansion in t:

$$g(t,x)-g(-t,x)=0(t^\infty) \tag{$**$}$$

He defined two operators L_1, L_2 satisfying $(*)$ and $(**)$ to be equivalent if L_2 can be obtained from L_1 upto smooth multiplier by a germ of diffeomorphism $(t,x)\to(\tilde{t},\tilde{x})$ near the origin where $\tilde{t}$ and $\tilde{x}$ are odd and even, respectively, in t to infinite order.

Let $\mathcal{L}$ be the set of the equivalent classes of vector fields satisfying $(*)$ and $(**)$. The main result of [4] is that the equivalent classes in $\mathcal{L}$ can be characterized by a certain equivalent classes of germs of diffeomorphisms $k:\mathbf{R}\to\mathbf{R}$ with $k(0)=0$, $k'>0$.

In this note we shall generalize the situation one step further and thus we consider germs of generalized Mizohata type vector fields of the forms

$$L=\frac{\partial}{\partial t}-it^k g(t,x)\frac{\partial}{\partial x},\quad \operatorname{Re} g(0,0)\neq 0,$$

where $k=2^n-1$ (n : positive integer) and where $g(t,x)$ has Taylor expansion

$$g(t,x)\sim a_0(x)+a_{k+1}(x)t^{k+1}+a_{2(k+1)}t^{2(k+1)}+\cdots\cdots,$$

and shall prove that the same results parallel to the ones in [4] can be obtained with respect to the above vectorfield L.

1. The associated pair of elliptic operators

In the following we shall systematically work with *germs* of smooth functions, *germs* of vector fields, and *germs* of diffeomorphisms, all defined near

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the origin of $\mathbf{R}^2$. The diffeomorphisms will map the origin into itself.

We consider (a germ of) a generalized Mizohata type operator L of the form

$$L=\frac{\partial}{\partial t}-it^k g(t,x)\frac{\partial}{\partial x},\quad \operatorname{Re} g(0,0)\neq 0 \tag{1}$$

with $k=2^n-1$ (n : positive integer), where $g(t,x)$ has a Taylor expansion in t

$$g(t,x)\sim a_0(x)+a_{k+1}(x)t^{k+1}+a_{2(k+1)}(x)t^{2(k+1)}+\cdots \tag{2}$$

when a function $g(t,x)$ satisfies (2) we shall say that $g(t,x)$ *is a function of* t^{k+1} *to infinite order.*

We note that (2) is equivalent to the fact

$$g(t,x)-g(-t,x)=0(t^\infty) \tag{3}$$

$$g_i(s,x)-g_i(-s,x)=0(s^\infty)\quad (i=1,2\ldots,n) \tag{4}$$

where $g_i(s,x)=g(s^{\frac{1}{2^i}},x)$.

MAIN THEOREM. *Let* $k:(t,x)\to(\tilde t(t,x),\tilde x(t,x))$ *be a diffeomorphism such that* $t\geq 0$ *iff* $\tilde t\geq 0$. *We assume that* $\tilde x(t,x)$ *is a function of* t^{k+1} *to infinite order. Then* L *transforms into a nonvanishing factor times*

$$\tilde L=\frac{\partial}{\partial t}-it^k\tilde g(t,\tilde x)\frac{\partial}{\partial\tilde x}.$$

Moreover, L *transforms into a nonvanishing factor times* $\tilde L$ *where* $\tilde g$ *satisfies* (2) *in* $\tilde t$ *if and only if* $\frac{\tilde t}{t}$ *and* $\tilde x$ *are functions of* t^{k+1} *to infinite order.*

Proof. Since $\frac{\partial}{\partial x}=\frac{\partial}{\partial t}\frac{\partial\tilde t}{\partial t}+\frac{\partial}{\partial\tilde x}\frac{\partial\tilde x}{\partial t}$ and $\frac{\partial}{\partial x}=\frac{\partial}{\partial\tilde t}\frac{\partial\tilde t}{\partial x}+\frac{\partial}{\partial\tilde x}\frac{\partial\tilde x}{\partial x}$, we have from $L=\frac{\partial}{\partial t}-it^k g(t,x)\frac{\partial}{\partial x}$

$$L=\left(\frac{\partial}{\partial\tilde t}\frac{\partial\tilde t}{\partial t}+\frac{\partial}{\partial\tilde x}\frac{\partial x}{\partial t}\right)-it^k g(t,x)\left(\frac{\partial}{\partial\tilde t}\frac{\partial\tilde t}{\partial x}+\frac{\partial}{\partial\tilde x}\frac{\partial x}{\partial x}\right)$$

$$=\left(\frac{\partial\tilde t}{\partial t}-ix^k g(t,x)\frac{\partial\tilde t}{\partial x}\right)\frac{\partial}{\partial t}+\left(\frac{\partial\tilde x}{\partial t}-it^k g(t,x)\right)\frac{\partial}{\partial\tilde x} \tag{5}$$

We note that k is a diffeomorphism such that $t\geq 0$ iff $t\geq 0$. Hence k is a diffeomorphism from $t=0$ to $t=0$. Thus $\frac{\partial t}{\partial t}\neq 0$. Therefore

$$\tilde L=\left(\frac{\partial\tilde t}{\partial t}-it^k g(t,x)\frac{\partial\tilde t}{\partial x}\right)\left|\frac{\partial}{\partial t}+\frac{\frac{\partial\tilde x}{\partial t}-it^k g(t,x)\frac{\partial\tilde x}{\partial x}}{\frac{\partial\tilde t}{\partial t}-it^k g(t,x)\frac{\partial\tilde t}{\partial x}}\frac{\partial}{\partial\tilde x}\right|$$

Now since $t(0, \tilde{x})=0$, $t(\tilde{t}, \tilde{x})=\tilde{t}p(\tilde{t}, \tilde{x})$ for a suitable smooth function $p(\tilde{t}, \tilde{x})$. Moreover, since $\tilde{x}=\tilde{x}(t, x)$ satisfies (2), $\frac{\partial \tilde{x}}{\partial t}=t^k g(t, x)=t^k r(t, \tilde{x})$ for a suitable smooth function $r(\tilde{t}, \tilde{x})$. Therefore, for suitable $\tilde{f}(\tilde{t}, \tilde{x})$ and $\tilde{g}(\tilde{t}, \tilde{x})$,

$$\tilde{L}=\tilde{f}(\tilde{t}, \tilde{x})\left(\frac{\partial}{\partial t}+i\tilde{t}^k \tilde{g}(\tilde{t}, \tilde{x})\frac{\partial}{\partial \tilde{x}}\right).$$

Now we shall prove the latter part of the theorem. The case for $n=1$ is proven by Sjöstrand in [4]. We shall prove for the case of $n=2$ and thus $k=3$. That $g(t, x)$ satisfies (2) for $k=3$ is equivalent to (3) and (4) for $i=1$.

Let τ be the reflection $\tau: (t, x)\to(-t, x)$. Then setting $s=-t$ in (5), we have

$$\tau_* L=-\frac{\partial}{\partial s}+is^k g(-s, x)\frac{\partial}{\partial x}.$$

If $g(t, x)$ satisfies (3), $g(s, x)-g(-s, x)=0(s^\infty)$. Setting $s=t$, we have

$$\tau_* L=-\left(\frac{\partial}{\partial t}-it^k\{g(t, x)+0(t^\infty)\}\frac{\partial}{\partial x}\right)$$
$$=-L+it^k 0(t^\infty)\frac{\partial}{\partial x}$$

and hence $\tau_* L=-L+0(t^\infty)$. (6)

Conversely, if (6) holds, we have (3). Therefore (3) is equivalent to (6).

Similar argument shows that (4) for $i=1$ is equivalent to

$$\tau_* L_1=-L_1+0(r^\infty)$$

when we set $L_1=\frac{\partial}{\partial r}-irg_1(r, x)$.

Sufficiency

We assume that $\frac{\tilde{t}}{t}$ is a function of t^4 to infinite order. Now if $k(t, x)=(t, \tilde{x})$, then

$$k(-t, x)=(-\tilde{t}+0(t^\infty), \tilde{x}+0(t^\infty)).$$

In fact, $\tilde{t}=\tilde{t}(t, x)$ is odd toinfinite order in t and hence

$$\tilde{t}(t, x)+\tilde{t}(-\tilde{t}, x)=0(t^\infty).$$

That is, $t(-t, x)=-t(t, x)+0(t^\infty)$.

On the while, $\tilde{x}=\tilde{x}(t, x)$ is a function of t^4 to infinite order and hence it is even to infinite order in t. Therefore,

$$\tilde{x}(t, x)-\tilde{x}(-t, x)=0(t^\infty).$$

That is, $\tilde{x}(-t, x)=\tilde{x}(t, x)+0(t^\infty)$.

Thus $k\tau k^{-1}=\tau+0(t^\infty)$ since $k(0(t^\infty))=0(t^\infty)$, and from

$$(k\tau k^{-1})_*\tilde{L}=k_*\tau_*(k_*^{-1}\tilde{L})=k_*\tau_*L$$
$$=-k_*(L+0(t^\infty))=-\tilde{L}+0(\tilde{t}^\infty)$$

it follows that

$$\tau_*\tilde{L}=-\tilde{L}+0(t^\infty).$$

This is equivalent to the fact

$$\tilde{g}(\tilde{t},\tilde{x})=\tilde{g}(-\tilde{t},\tilde{x})+0(\tilde{t}^\infty).$$

Now we consider diffeomorphisms from the upper half plane to the upper half plane such that

$$\phi:(t,x)\rightarrow(s,x) \text{ where } s=t^2$$
$$\psi:(\tilde{t},\tilde{x})\rightarrow(\tilde{s},\tilde{x}) \text{ where } \tilde{s}=\tilde{t}^2.$$

We consider the diffeomorphism $\mu=\psi k\phi^{-1}:(s,x)\rightarrow(\tilde{s},\tilde{x})$ from $s\geq 0$ to $\tilde{s}\geq 0$. This diffeomorphism can naturally be extended to the lower half plane such that

$$\mu:(s,x)\rightarrow(\tilde{t},\tilde{x}) \text{ by } \tilde{t}=-(-\tilde{t}) \text{ for } s<0.$$

In fact, $\tilde{t}\sim t(b_1(x)+b_5(x)t^4+b_9(x)t^8+\cdots)$.
Therefore $\tilde{t}^2\sim t^2(b_1(x)+b_5(x)t^4+b_9(x)t^\infty+\cdots)^2$,
which says

$$\tilde{s}\sim s(b_1(x)+b_5(x)s^2+b_9(x)s^4+\cdots)^2$$

can be extended as a smooth function for $s<0$. Similarly $\tilde{x}=\tilde{x}(t^4,x)=\tilde{x}(s^2,x)$ can bedefined as a smooth function for $s<0$.

Now we notice that the diffeomorphism $\mu:(s,x)\rightarrow(\tilde{s},\tilde{x})$ satisfies that $\tilde{s}$ is odd and $\tilde{x}$ is even in s to infinite order. Notice also that

$$\phi_*L=\left(\frac{\partial}{\partial s}-is\frac{g_1(s,x)}{2}\frac{\partial}{\partial x}\right)\sqrt{s},$$
$$\psi_*\tilde{L}=\left(\frac{\partial}{\partial\tilde{s}}-i\tilde{s}\frac{\tilde{g}_1(\tilde{s},\tilde{x})}{2}\frac{\partial}{\partial\tilde{x}}\right)\sqrt{\tilde{s}}$$

and hence

$$\mu_*\left(\frac{\partial}{\partial s}-is\frac{g_1(s,x)}{2}\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial\tilde{s}}-i\tilde{s}\frac{\tilde{g}_1(\tilde{s},\tilde{x})}{2}\frac{\partial}{\partial\tilde{x}}.$$

If $g_1(s,x)$ satisfies (4), then $g_1(s,x)$ is even in s. Therefore from the Lemma 1.1 in [4], it follows that $g_1(\tilde{s},x)$ is even in $\tilde{s}$ to infinite order.

Necessity

We first remark that under the assumption (3) : *If* $\sigma:(t,x)\rightarrow(\tilde{t},\tilde{x})$ *is a diffeomorphism such that* $\sigma|_{t=0}=\text{id.}$ *and* $\sigma_*L=f(t,x)L+0(t^\infty)$ *for a suitable smooth function* f, *then* $\tilde{t}=\pm t+0(t^\infty)$ *and*

$$\tilde{x}=x+0(t^\infty) \tag{7}$$

To see (7) we note that the problem

$$Lu=0(t^{\infty}),\quad u(0,x)=x$$

admits a solution $u(t,x)$ with the Taylor expansion

$$u(t,x)\sim x+\frac{it^4}{4}g(0,x)+c_8(x)t^8+\cdots$$

In fact, $u(0,x)=x$, and

$$\begin{aligned}Lu&=\left(\frac{\partial}{\partial t}-it^3g(t,x)\frac{\partial}{\partial x}\right)u\\&=it^3g(0,x)+8c_8(x)t^7+\cdots\\&=it^3(g(0,x)+a_4(x)t^4+a_8(x)t^8+\cdots)\\&\quad\left[1+\frac{it^4}{4}g'(0,x)+c_8'(x)t^8+\cdots\right].\end{aligned}$$

Therefore we may determine $c_{4n}(x)$ successively so that $Lu=0(t^{\infty})$.

Now if $v=u-\sigma^*u$, we have

$$\begin{aligned}Lu&=Lu-L\sigma^*u=Lu-\sigma_*Lu\\&=Lu-(f(t,x)L+0(t_{\infty}))u=0(t^{\infty}),\\v(0,x)&=u(0,x)-(\sigma^*u)(0,x)\\&=u(0,x)-u(\sigma(0,x))=u(0,x)-u(0,x)=0.\end{aligned}$$

Thus by the uniqueness of the solution, we have

$$v=0(t^{\infty}),\qquad\text{or}\qquad\sigma^*u=u+0(t^{\infty}).$$

Thus we have

$$\begin{aligned}&x+\frac{it^4}{4}g(0,x)+c_8(x)t^8+\cdots+0(t^{\infty})\\&=\tilde{x}+\frac{it^4}{4}g(0,\tilde{x})+c_8(\tilde{x})\tilde{t}^8+\cdots,\end{aligned}$$

whence $\tilde{x}=x+0(t^{\infty})$ and $\tilde{t}=\pm t+0(t^{\infty})$ since $\operatorname{Re} g(0,0)\neq 0$. Thus (5) is deduced.

Now we assume that $\tilde{g}(\tilde{t},\tilde{x})$ satisfies (3) and (4) for $i=1$. That $\tilde{g}(\tilde{t},\tilde{x})-\tilde{g}(-\tilde{t},\tilde{x})=0(t^{\infty})$ implies $\tau_*\tilde{L}=-\tilde{L}+0(\tilde{t}^{\infty})$. On the other hand, we consider the map $k\tau k^{-1}$ and assume that $k(t,x)=(\tilde{t},\tilde{x})$.

Since
$$\begin{aligned}(k\tau k^{-1})_*\tilde{L}&=k_*\tau_*(k_*^{-1}\tilde{L})=k_*\tau_*L\\&=-k_*(L+0(t^{\infty}))=-\tilde{L}+0(t^{\infty}),\end{aligned}$$

by the preceding remarks applied for $\sigma=k\tau k^{-1}$, we have

$$(k\tau k^{-1}):(\tilde{t},\tilde{x})\to(\pm\tilde{t}+0(t^{\infty}),\tilde{x}+0(\tilde{t}^{\infty})).$$

Therefore

$$\begin{aligned}k(-\tilde{t},\tilde{x})&=k\tau(t,x)=(k\tau k^{-1})(\tilde{t},\tilde{x})\\&=(\tilde{t}+0(\tilde{t}^{\infty}),\tilde{x}+0(\tilde{t}^{\infty}))\text{ or }(-\tilde{t}+0(\tilde{t}^{\infty}),\tilde{x}+0(\tilde{t}^{\infty})).\end{aligned}$$

Since $k(\tilde{t}, x)=(\tilde{t}, x)$ and k is a diffeomorphism, this implies

$$k(-\tilde{t}, \tilde{x})=(-\tilde{t}+0(t^{\infty}), \tilde{x}+0(\tilde{t}^{\infty})).$$

Thus $(k\tau k^{-1}) : (\tilde{t}, x)\to(-\tilde{t}+0(\tilde{t}^{\infty}), \tilde{x}+0(\tilde{t}^{\infty}))$.
Therefore $k\tau k^{-1}=\tau+0(\tilde{t}^{\infty})$.

The commutative diagram

$$\begin{array}{ccc} (t, x) & \xleftarrow{k^{-1}} & (\tilde{t}, \tilde{x}) \\ \downarrow \tau & & \downarrow \tau \\ (-t, x) & \xrightarrow{k} & (-\tilde{t}+0(\tilde{t}^{\infty}),\ \tilde{x}+0(\tilde{t}^{\infty})) \end{array}$$

implies that $\tilde{t}$ is odd and $\tilde{x}$ is even in $\tilde{t}$ to infinite order.

Finally we focus our attention to the condition (4) for $i=1$. Thus we assume that $\tilde{g}_1(\tilde{s}, \tilde{t})-\tilde{g}_1(-\tilde{s}, \tilde{x})=0(\tilde{s}^{\infty})$. Since $\tilde{t}$ is odd and $\tilde{x}$ is even in t to infinite order by the preceding argument, we have

$$\tilde{t}\sim b_1(x)t+b_3(x)t^3+b_5(x)t^5++\cdots.$$

Hence $$\tilde{t}^2\sim d_1(x)t^2+d_4(x)t^4+d_6(x)t^6+\cdots.$$

That is, $$\tilde{t}\sim d_1(x)s+d_4(x)s^2+d_6(x)s^3+\cdots. \qquad (9)$$

where $\tilde{s}=\tilde{t}^2$ and $s=t^2$. Since $\tilde{x}$ is even in t to infinite order,

$$\tilde{x}=e_0(x)+e_2(x)s+e_4(x)s^2+\cdots. \qquad (10)$$

By (9) and (10) the diffeomorphism $\mu : (s, x)\to(\tilde{s}, \tilde{x})$ defined in the upper half plane $s\geq 0$ can be extended as a diffeomorphism for the full neighborhood of the origin. Also by this extended diffeomorphism μ, $L_1=\frac{\partial}{\partial s}-is\frac{g_1(s, x)}{2}\frac{\partial}{\partial x}$ transforms into a smooth multiple of $L_1=\frac{\partial}{\partial \tilde{s}}-i\tilde{s}\frac{\tilde{g}_1(\tilde{s}, \tilde{x})}{2}\frac{\partial}{\partial \tilde{x}}$. Therefore the same argument we used in the above applied for $L_1=\frac{\partial}{\partial s}-is\frac{g_1(s, x)}{2}\frac{\partial}{\partial x}$ and $\tilde{L}_1=\frac{\partial}{\partial \tilde{s}}-i\tilde{s}\frac{\tilde{g}_1(\tilde{s}, \tilde{x})}{2}\frac{\partial}{\partial \tilde{x}}$ concludes that $\tilde{s}$ is odd and$x$ is even in s to infinite order (*cf.* also Lemma 1.1 in [4]). Thus

$$\tilde{s}\sim d_1(x)s+d_3(x)s^3+\cdots,$$
$$\tilde{x}\sim e_0(x)+e_4(x)s^2+e_8(x)s^4+\cdots.$$

That is,

$$\begin{aligned} \tilde{t}^2 &\sim d_1(x)t^2+d_6(x)t^6+\cdots \\ \tilde{x} &\sim e_0(x)+e_4(x)t^4+e_8(x)t^8+\cdots \end{aligned} \qquad (11)$$

(11) is possible only when

$$\tilde{t}\sim b_1(x)t+b_5(x)t^5+b_9(x)t^9+\cdots.$$

Therefore $\frac{\tilde{t}}{t}$ and $\tilde{x}$ are functionsof t^4 to infinite order. This completes our proof for $k=3$.

For the general case for $k=2^n-1$, we can use the above arguments recursively. (Q. E. D.)

The map k which satisfies latter part of the theorem is called an *admissible transformation.*

Consider the map $(t, x) \to (y, x)$, $y=\frac{1}{k+1}t^{k+1}$. Then if L is given by (1) and (2), we see that

$$L|_{t\geq 0}=t^k L_+, \quad L|_{t\leq 0}=t^k L_-$$

Where

$$L_\pm=\frac{\partial}{\partial y}-ig\,(\pm[(k+1)y]^{\frac{1}{k+1}}, x)\frac{\partial}{\partial x} \tag{12}$$

are elliptic with smooth coefficients in $y\geq 0$ and agree to infinite order on $y=0$.

Conversely, if

$$L_\pm=\frac{\partial}{\partial y}-ig_\pm(y, x)\frac{\partial}{\partial x}$$

is such a pair of elliptic operators which agree to infinite order at $y=0$, then we can define an operator L satisfying (1) and (2) by setting

$$g(t, x)=g_\pm\left(\frac{1}{k+1}t^{k+1},\ x\right) \text{ for } t\gtrless 0 \tag{13}$$

We call (L_+, L_-) an *admissible pair*. An *admissible pair of diffeomorphisms* (k_+, k_-) is a pair of diffeomorphisms $k_\pm$: $\{y\geq 0\}\to\{y\geq 0\}$ smooth up to the x-axis and conserving the x-axis which agree to infinite order. The admissible pair $(\tilde{L}_+, \tilde{L}_-)$ is obtained from (L_+, L_-) by an admissible transformaton if $\tilde{L}_\pm=f_\pm((k_\pm)_* L_\pm)$ for suitable nonvanishing factors f, where (k_+, k_-) is an admissible pair of diffeomorphisms.

It can be easily proved that the map $(t, x) \to \left(\frac{t^{k+1}}{k+1}, x\right)$ gives rise to a bijection between the set of admissible transformation and the set of admissible pairs (k_+, k_-) of diffeomorphisms.

2. Applications

Once the main theorem in 1 is proved, the theorems in Sjöstrand [4] can be generalized to our operator L satisfying (1) and (2). The proofs are all exactly same as in [4]. For example, we have

THEOREM 2.1. *If L satisfies* (1) *and* (2), *there exists an admissible transformation taking L into*

$$\tilde{L}=\frac{\partial}{\partial t}-it^k(1+\rho(t,\ x))\frac{\partial}{\partial x} \tag{14}$$

where ρ vanishes for $t\leq 0$.

Following Sjöstrand we also define for two (germs of) diffeomorphisms with positive derivatives $k_1, k_2 : R\to R$ with $k_i(0)=0(i=1, 2)$ to *be equivalent* if $k_1=h\circ k_2\circ g$ and h, g are analytic diffeomorphisms with positive derivatives.

Let $\mathcal{M}$ be the set of such equivalent class of diffeomorphisms $k : R\to R$ with $k(0)=0$, $k'>0$. If L_1 and L_2 are two operators satisfying (1) and (2) we say that they *are equivalent* if one obtained from the other by an admissible transformation.

Let $\mathcal{L}$ be the set of such equivalent classes. We have

THEOREM 2.2. *There is a bijective map between $\mathcal{L}$ and $\mathcal{M}$.*

The proof is again same as the one in [4]. We omit the proof.

Consequences of the Theorem 2.2 are as follows. (*cf.* [2] and [4])

COROLLARY 1. *Let L satisfy* (1) *and* (2). *Let* (*the germ*) $u\in C^1(\mathbf{R}^2)$ *satisfying* $du(0, 0)\neq 0$ *and* $Lu=0$. *Then L is equivalent to*

$$\frac{\partial}{\partial t}-it^k\frac{\partial}{\partial x}.$$

COROLLARY 2. *There exists L satisfying* (1) *and* (2) *such that* $u\in C^1$, $Lu=0$ *implies* $u=const.$

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