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ON CONNECTIONS WITH TORSIONS AND THOSE CURVATURE TENSORS IN RIEMANNIAN MANIFOLDS II *

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I. A characterization of a Kaehlerian manifold by a curvature tensor formed with an F-connection with torsion.

§ 1. *Preliminaries.*

We consider a Kaehlerian manifold M of real $2n$ dimensions ($n\geqq 2$) covered by a coordinate neighborhoods $\{U\,;\,x^h\}$ and denote by g_{ji} and $F_i{}^h$ components of the Hermitian metric tensor and those of the Kaehlerian structure tensor of M respectively, where, here and in the sequel, the indices h, i, $j,\cdots$ run over the range $\{1,2,\cdots,2n\}$.

Let $'D$ be an affine connection with torsion in a Kaehlerian manifold M. We denote by $'\Gamma_{kj}{}^h$ the components of the connection $'D$ and by $'D_j$ the operator of covariant differentiation with respect to $'\Gamma_{kj}{}^h$.

If the affine connection $'D$ satisfies

$$(1.1)\qquad 'D_k g_{ji}=-2p_k g_{ji},$$

$$(1.2)\qquad 'D_k F_{ji}=-2p_k F_{ji}\quad (\text{or } 'D_k F_j{}^h=0),$$

$$(1.3)\qquad '\Gamma_{ji}{}^h-'\Gamma_{ij}{}^h=-2F_{ji}q^h,$$

where $F_{ji}=g_{ih}F_j{}^h$, for a certain non zero covector field p_k and a vector field q^h, then $'D$ is called a *complex Weyl-Hlavaty connection* [4].

Solving (1.1) and (1.3) with respect to $'\Gamma_{ji}{}^h$, it is easily obtained that

$$(1.4)\qquad '\Gamma_{ji}{}^h=\left\{{h \atop j\ i}\right\}+\delta_j^h p_i+\delta_i^h p_j-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h,$$

where $p^h=p_t g^{th}$, $q^h=q_t g^{th}$ and $\left\{{h \atop j\ i}\right\}$ are the Christoffel symbols of M.

Computing $'D_k F_{ji}$ and taking account of (1.4), we obtain

$$(n-1)(p_t F_i{}^t+q_i)=0,$$

from which

$$(1.5)\qquad q_i=-p_t F_i{}^t,\quad p_i=q_t F_i{}^t.$$

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In a previous paper [1], we considered an affine connection D whose components Γ_{ji}^h are related to those of complex Weyl-Hlavaty connection $'D$ by

$$(1.6)\qquad \Gamma_{ji}^h = {}'\Gamma_{ji}^h - p_j\delta_i^h,$$

that is,

$$(1.7)\qquad \Gamma_{ji}^h = \{{}_j{}^h{}_i\} + \delta_j^h p_i - g_{ji}p^h + F_j^h q_i + F_i^h q_j - F_{ji}q^h$$

in a Kaehlerian manifold M.
(Such a connetion D is called "a semi-symmetric metric F-connection" in [5].)
Denoting by R_{kji}^h the curvature tensor of Γ_{ji}^h, we obtain

$$(1.8)\qquad \begin{aligned} R_{kji}^h = K_{kji}^h - \delta_k^h p_{ji} + \delta_j^h p_{ki} - p_k^h g_{ji} + p_j^h g_{ki} - F_k^h q_{ji} \\ + F_j^h q_{ki} - q_k^h F_{ji} + q_j^h F_{ki} - \alpha_{kj}F_i^h - F_{kj}\beta_i^h, \end{aligned}$$

where K_{kji}^h is the Riemannian curvature tensor of M,

$$(1.9)\qquad p_{ji} = \nabla_j p_i - p_j p_i + q_j q_i + \frac{1}{2}\lambda g_{ji},$$

$$(1.10)\qquad q_{ji} = \nabla_j q_i - p_j q_i - q_j p_i + \frac{1}{2}\lambda F_{ji},$$

λ being defined by $\lambda = p_t p^t = q_t q^t$,

$$(1.11)\qquad \alpha_{ji} = -(\nabla_j q_i - \nabla_i q_j),\quad \beta_{ji} = 2(p_j q_i - q_j p_i)$$

and $p_k^h = p_{kt}g^{th}$, $q_k^h = q_{kt}g^{th}$, $\beta_i^h = \beta_{it}g^{th}$.

The following relations are easily checked.

$$(1.12)\qquad p_{ji} = q_{jt}F_i^t,\quad q_{ji} = -p_{jt}F_i^t,$$

$$(1.13)\qquad \alpha_{ji} = -(q_{ji} - q_{ij} - \lambda F_{ji}).$$

§2. *A certain F-connection which is closely related to the complex Weyl-Hlavaty connection.*

In this section, we use the fact that if

$$\Gamma_{ji}^h = \{{}_j{}^h{}_i\} + T_{ji}^h,$$

T_{ji}^h being a tensor field of type $(1,2)$, then the curvature tensor R_{kji}^h formed with Γ_{ji}^h is given by

$$(2.1)\qquad R_{kji}^h = K_{kji}^h + \nabla_k T_{ji}^h - \nabla_j T_{ki}^h + T_{kt}^h T_{ji}^t - T_{jt}^h T_{ki}^t.$$

In this place, we want to seek out an F-connection $^*D$ in a Kaehlerian manifold M such that if the curvature tensor formed with the components of

$*D$ vanishes then M is of constant holomorphic sectional curvature.

For this aim, it is desirable that the tensor p_{ji} in (1. 8) is symmetric. By this reason, we firstly put

$$(2.2) \qquad {}_{(1)}\Gamma_{ji}^{h}=\Gamma_{ji}^{h}-p_{j}F_{i}^{h},$$

where Γ_{ji}^h are the components of a metric connetion D defined by (1. 7).

Denoting by ${}_{(1)}D_j$, the operator of covariant differentiation with respect to ${}_{(1)}\Gamma_{ji}^h$, we find that ${}_{(1)}D_j$ is an F-connection, that is, ${}_{(1)}D_kF_j^i=0$ and the curvature tensor ${}_{(1)}R_{kji}^h$ of ${}_{(1)}\Gamma_{ji}^h$ is of the form:

$$(2.3) \qquad {}_{(1)}R_{kji}^{h}=R_{kji}^{h}-(p_{kj}-p_{jk})F_{i}^{h},$$

where R_{kji}^h is the curvature tensor of Γ_{ji}^h defined by (1. 7).

Secondly, it is desirable for our aim that the expecting F-connection annihilates the terms $-F_k^hq_{ji}+F_j^hq_{ki}$ in (1. 8). By this reason, we put

$$(2.4) \qquad {}_{(2)}\Gamma_{ji}^{h}={}_{(1)}\Gamma_{ji}^{h}-F_{j}^{h}q_{i}.$$

Then the curvature tensor ${}_{(2)}R_{kji}^h$ of ${}_{(2)}\Gamma_{ji}^h$ is of the form

$$(2.5) \qquad \begin{aligned}{}_{(2)}R_{kji}^{h}={}&{}_{(1)}R_{kji}^{h}-F_{j}^{h}({}_{(1)}D_{k}q_{i})+F_{k}^{h}({}_{(1)}D_{j}q_{i})\\ &-q_{i}({}_{(1)}\Gamma_{kj}^{t}-{}_{(1)}\Gamma_{jk}^{t})F_{t}^{h}\\ &+p_{j}q_{i}F_{k}^{h}-p_{k}q_{i}F_{j}^{h}.\end{aligned}$$

From (1. 6) and (2. 2), we easily obtain

$$(2.6) \qquad {}_{(1)}D_{k}q_{i}=\nabla_{k}q_{i}-q_{k}p_{i}-p_{k}q_{i}-p_{i}q_{k}+\lambda F_{ki}+p_{k}p_{i},$$

$$(2.7) \qquad {}_{(1)}\Gamma_{kj}^{h}-{}_{(1)}\Gamma_{jk}^{h}=-2F_{kj}q^{h}-\delta_{k}^{h}p_{j}+\delta_{j}^{h}p_{k}-p_{k}F_{j}^{h}-p_{j}F_{k}^{h}.$$

Finally, it is also desirable that the expecting F-connection annihilates the terms $-\delta_k^hp_{ji}+\delta_j^hp_{ki}$ in (1. 8), so we put

$$(2.8) \qquad {}^{*}\Gamma_{ji}^{h}={}_{(2)}\Gamma_{ji}^{h}-\delta_{j}^{h}p_{i}.$$

Denoting by ${}_{(2)}D_k$ the operator of covariant differentiation with respect to ${}_{(2)}\Gamma_{ji}^h$, we find that the curvature tensor ${}^*R_{kji}^h$ of ${}^*\Gamma_{ji}^h$ is of the form:

$$(2.9) \qquad \begin{aligned}{}^{*}R_{kji}^{h}={}&{}_{(2)}R_{kji}^{h}-\delta_{j}^{h}({}_{(2)}D_{k}p_{i}+p_{k}p_{i})+\delta_{k}^{h}({}_{(2)}D_{j}p_{i}+p_{j}p_{i})\\ &-p_{i}({}_{(2)}\Gamma_{kj}^{h}-{}_{(2)}\Gamma_{jk}^{h}).\end{aligned}$$

From (2. 4), (2. 6) and (2. 7), we easily obtain

$$(2.10) \qquad {}_{(2)}D_{k}p_{i}=\nabla_{k}p_{i}-p_{k}p_{i}+\lambda g_{ki}+q_{i}q_{k}-p_{k}q_{i},$$

$$(2.11) \qquad {}_{(2)}\Gamma_{ji}^{h}-{}_{(2)}\Gamma_{ij}^{h}=-2F_{ji}q^{h}-\delta_{i}^{h}p_{j}+\delta_{j}^{h}p_{i}-p_{j}F_{i}^{h}+p_{i}F_{j}^{h}-F_{j}^{h}q_{i}+F_{i}^{h}q_{j}.$$

In the sequel, we introduced an F-connection $*D$ whose components ${}^*\Gamma_{ji}^h$

are given by

$$(2.12)\qquad {}^*\Gamma_{ji}^{\ \ h} = {}'\Gamma_{ji}^{\ \ h} - p_j\delta_i^h - p_jF_i^{\ h} - F_j^{\ h}q_i - \delta_j^h p_i,$$

where ${}'\Gamma_{ji}^{\ \ h}$ are the components of the complex Weyl-Hlavaty connection ${}'D$. Then (2.12) is written as

$$(2.13)\qquad {}^*\Gamma_{ji}^{\ \ h} = \left\{ {h \atop j\ i} \right\} + q_jF_i^{\ h} - p_jF_i^{\ h} - g_{ji}p^h - F_{ji}q^h$$

by the help of (1.4).

Substituting (2.12) into (1.1), (1.2) and (1.3) and denoting by ${}^*D_k$ the covariant differentiation with respect to ${}^*\Gamma_{ji}^{\ \ h}$, we find

$$(2.14)\qquad {}^*D_kg_{ji} = g_{kj}p_i + g_{ki}p_j + F_{kj}q_i + F_{ki}q_j,$$

$$(2.15)\qquad {}^*D_kF_{ji} = g_{kj}q_i - g_{ki}q_j - F_{kj}p_i + F_{ki}p_j,\quad (\text{or } {}^*D_kF_j^{\ h} = 0).$$

In this case, we obtain

$$(2.16)\qquad {}^*\Gamma_{ji}^{\ \ h} - {}^*\Gamma_{ij}^{\ \ h} = -2F_{ji}q^h + (q_j - p_j)F_i^{\ h} - (q_i - p_i)F_j^{\ h},$$

and we call ${}^*D$ an *F-connection which is closely related to the complex Weyl-Hlavaty connection.*

Substituting (1.8), (2.3) and (2.5) into (2.9) successively, we obtain

$$(2.17)\qquad \begin{aligned} {}^*R_{kji}^{\ \ \ h} = K_{kji}^{\ \ \ h} &- g_{ji}(p_k^{\ h} - \tfrac{1}{2}\lambda\delta_k^h) + g_{ki}(p_j^{\ h} - \tfrac{1}{2}\lambda\delta_j^h) \\ &- F_{ji}(q_k^{\ h} - \tfrac{1}{2}\lambda F_k^{\ h}) + F_{ki}(q_j^{\ h} - \tfrac{1}{2}\lambda F_j^{\ h}) \\ &+ (q_{kj} - q_{jk} - \lambda F_{kj})F_i^{\ h} - (p_{kj} - p_{jk})F_i^{\ h}, \end{aligned}$$

where p_{ji} and q_{ji} are defind by (1.9) and (1.10) respectively.

§ 3. *A characterization of a Kaehlerian manifold by the curvature tensor* ${}^*R_{kji}^{\ \ \ h}$ *defined by* (2.17).

In this section, we prove the following

THEOREM 3.1. *Let a real 2n-dimensional Kaehlerian manifold $M(n\geqq 2)$ admits the complex Weyl-Hlavaty connection. If the curvature tensor formed with the components of the connection ${}^*D$ defined by* (2.13) *vanishes locally, then M is of constant holomorphic sectional curvature locally, and vice-versa.*

Proof. Assuming that

$$(3.1)\qquad {}^*R_{kji}^{\ \ \ h} = 0,$$

we can write (2.17) in covariant form as

$$K_{kjih}=g_{ji}(p_{kh}-\frac{1}{2}\lambda g_{kh})-g_{ki}(p_{jh}-\frac{1}{2}\lambda g_{jh})+F_{ji}(q_{kh}-\frac{1}{2}\lambda F_{kh})$$

$$-F_{ki}(q_{jh}-\frac{1}{2}\lambda F_{jh})-(q_{kj}-q_{jk}-\lambda F_{kj})F_{ih}+(p_{kj}-p_{jk})F_{ih}. \tag{3.2}$$

For briefness, we now put

$$P_{ji}=p_{ji}-\frac{1}{2}\lambda g_{ji} \tag{3.3}$$

and

$$Q_{ji}=q_{ji}-\frac{1}{2}\lambda F_{ji}. \tag{3.4}$$

Then (3.2) becomes

$$K_{kjih}=g_{ji}P_{kh}-g_{ki}P_{jh}+F_{ji}Q_{kh}-F_{ki}Q_{jh}-(Q_{kj}-Q_{jk})F_{ih}+(P_{kj}-P_{jk})F_{ih}. \tag{3.5}$$

Transvecting (3.3) with $F_k{}^i$ and using (1.12), we find

$$P_{jt}F_k{}^t=-Q_{jk}, \qquad Q_{jt}F_k{}^t=P_{jk}. \tag{3.6}$$

Transvecting (3.5) with g^{kh}, taking account of (3.6), and denoting by K_{ji} the Ricci tensor of M, we obtain

$$K_{ji}=Pg_{ji}+QF_{ji}+P_{ji}-Q_{tj}F_i{}^t+P_{tj}F_i{}^t+Q_{ji}, \tag{3.7}$$

where we have put $P=P_t{}^t$ and $Q=Q_t{}^t$.

Transvecting (3.7) with $F_k{}^i$, and taking account of (3.6), we find

$$K_{jt}F_i{}^t=PF_{ij}+Qg_{ji}-Q_{ji}+Q_{ij}-P_{ij}+P_{ji}. \tag{3.8}$$

Substituting (3.8) into the well known equation $K_{jt}F_i{}^t+K_{it}F_j{}^t=0$, we obtain

$$Q=0. \tag{3.9}$$

Substituting (3.5) into the well known identity $K_{kjih}-K_{ihkj}=0$ and transvecting it with g^{ih}, we obtain

$$P_{kj}-P_{jk}=0 \tag{3.10}$$

by virtue of (3.6) and (3.9).

Substituting (3.5) into the identity $K_{kjih}=K_{ihkj}$ and transvecting it with F^{kh}, we find

$$PF_{ji}+(2n-1)Q_{ij}-Q_{ji}=0. \tag{3.11}$$

Taking the symmetric part of (3.11), we obtain

$$Q_{ji}+Q_{ij}=0. \tag{3.12}$$

Substituting (3.12) into (3.11), we find

$$Q_{ji}=\frac{P}{2n}F_{ji}. \tag{3.13}$$

Substituting (3.13) into (3.6), we obtain

$$P_{ji}=\frac{P}{2n}g_{ji}. \tag{3.14}$$

Substituting (3.9), (3.13) and (3.14) into (3.7), we obtain

$$K_{ji}=(1+\frac{1}{n})Pg_{ji}, \tag{3.15}$$

and from which, the scalar curvature K of M is given by

$$K=2(n+1)P. \tag{3.16}$$

Substituting (3.15) and (3.16) into the identity $\nabla_j K=2\nabla_t K_j^{\ t}$, we obtain $\nabla_j P=\frac{1}{n}(\nabla_t P)\delta_j^t$ and from which, $\nabla_j P=0$, that is, P=constant. Thus we obtain

$$P=\frac{K}{2(n+1)}, \qquad K=\text{constant}. \tag{3.17}$$

Substituting (3.13), (3.14) and (3.17) into (3.5), we find

$$K_{kji}^{\ \ \ h}=\frac{K}{2n(2n+1)}(g_{ji}\delta_k^h-g_{ki}\delta_j^h+F_{ji}F_k^{\ h}-F_{ki}F_j^{\ h}-2F_{kj}F_i^{\ h}), \tag{3.18}$$

K being a constant.

Conversely, if the Riemann-Christoffel curvature tensor of M is of the form (3.18), then we consider the integrability condition of the differential equations

$$\begin{aligned} &p_i=\nabla_i p, \\ &\nabla_j p_i=p_j p_i-q_j q_i+\frac{K}{2n(2n+1)}g_{ji}, \end{aligned} \tag{3.19}$$

where K is a constant and

$$q_j=-p_t F_j^{\ t}. \tag{3.20}$$

By a straightforward computation, we see that the following equation is satisfied

$$\nabla_k\nabla_j p_i-\nabla_j\nabla_k p_i=-K_{kji}^{\ \ \ t}p_t$$

by virtue of (3.18).

Therefore a gradient covector field p_i satisfying (3.19) is completely integrable locally. In this case, differentiating covariantly (3.20) and substituting (3.19) into it, we easily obtain

$$\nabla_j q_i = p_j q_i - q_j p_i + \frac{K}{2n(2n+1)} F_{ji}. \tag{3.21}$$

Substituting (3.19) and (3.21) into (1.9) and (1.10) respectively, we obtain

$$p_{ji} - \frac{\lambda}{2} g_{ji} = \frac{K}{2n(2n+1)} g_{ji}, \tag{3.22}$$

$$q_{ji} - \frac{\lambda}{2} F_{ji} = \frac{K}{2n(2n+1)} F_{ji}. \tag{3.23}$$

Substituting (3.22) and (3.23) into (2.17), we see that

$$*R_{kji}{}^h = 0$$

by virtue of (3.18). Thus the theorem 3.1 is completely proved. (cf. [6])

II. A characterization of a cosymplectic manifold by a curvature tensor formed with a φ-connection with torsion.

§1. *Preliminaries.*

We consider a cosymplectic manifold M of real $2n+1$ dimensions covered by a coordinate neighborhoods $\{U; y^h\}$ and denote by g_{ji}, $\varphi_j{}^h$, ξ^h and η_j components of the Grayan metric tensor, those of the cosymplectic structure tensor, those of the cosymplectic vector and those of the cosymplectic 1-form of M respectively, where, here and in the sequel, the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$.

We denote by $K_{kji}{}^h$, K_{ji} and K the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

Let $'D$ be an affine connection with torsion in a cosymplectic manifold M. We denote by $'\Gamma_{kj}{}^h$ the components of the connection $'D$ and by $'D_j$ the operator of covariant differentiation with respect to $'\Gamma_{kj}{}^h$.

If the affine connection $'D$ satisfies

$$'D_k g_{ji} = -2p_k \gamma_{ji}, \tag{1.1}$$

where $\gamma_{ji} = g_{ji} - \eta_j \eta_i$,

$$'D_k \varphi_{ji} = -2p_k \varphi_{ji} \quad (\text{or } 'D_k \varphi_j{}^h = 0), \tag{1.2}$$

$$'D_k \xi^h = 0 \qquad (\text{or } 'D_k \eta_j = 0), \tag{1.3}$$

(1.4) $$'\Gamma_{ji}^{h}-'\Gamma_{ij}^{h}=-2\varphi_{ji}u^{h},$$

for a certain non-zero covector field p_k and a vector field u^h, then $'D$ is called in [1] a *cosymplectic Weyl-Hlavaty connection.*

In a previous paper [1], we constructed an affiine connection D whose components Γ_{kj}^{h} are given by $'\Gamma_{kj}^{h}-p_k\gamma_j^{h}$, that is,

(1.5) $$\Gamma_{ji}^{h}=\{{}_{j}{}^{h}{}_{i}\}+p_i\gamma_j^{h}-p^h\gamma_{ji}+\varphi_j^{h}q_i+\varphi_i^{h}q_j-\varphi_{ji}q^h,$$

where $\{{}_{j}{}^{h}{}_{i}\}$ are the Christoffel symbols of M and

(1.6) $$q_i=-p_t\varphi_i^{t},\quad p_i=q_t\varphi_i^{t},\quad \gamma_j^{i}=\gamma_{jt}g^{ti}.$$

Denoting by R_{kji}^{h} the curvature teosor of Γ_{ji}^{h}, we obtain (cf. [1])

(1.7) $$R_{kji}^{h}=K_{kji}^{h}-\gamma_k^{h}p_{ji}+\gamma_j^{h}p_{ki}-p_k^{h}\gamma_{ji}+p_j^{h}\gamma_{ki}-\varphi_k^{h}q_{ji}$$
$$+\varphi_j^{h}q_{ki}-q_k^{h}\varphi_{ji}+q_j^{h}\varphi_{ki}-\alpha_{kj}\varphi_i^{h}+\varphi_{kj}\beta_i^{h},$$

where

(1.8) $$p_{ji}=\nabla_j p_i-p_jp_i+q_jq_i+\frac{1}{2}\lambda\gamma_{ji},$$

(1.9) $$q_{ji}=\nabla_j q_i-p_jq_i-q_jp_i+\frac{1}{2}\lambda\varphi_{ji},$$

λ being defined by $\lambda=p_tp^t=q_tq^t$,

(1.10) $$\alpha_{ji}=-(\nabla_j q_i-\nabla_i q_j),$$

(1.11) $$\beta_{ji}=2(p_jq_i-p_iq_j)$$

and $p_k^{h}=p_{kt}g^{th}$, $q_k^{h}=q_{kt}g^{th}$, $\beta_k^{h}=\beta_{kt}g^{th}$.

The following relations are easily checked.

(1.12) $$p_{ji}=q_{jt}\varphi_i^{t},\quad q_{ji}=-p_{jt}\varphi_i^{t},$$

(1.13) $$\alpha_{ji}=-(q_{ji}-q_{ij}-\lambda\varphi_{ji}).$$

§2. *A certain φ-connection which is closely related to the cosymplectic Weyl-Hlavaty connection.*

In this section, we want to seek out a φ-connection $^*D$ in a cosymplectic manifold M such that if the curvature tensor formed with the components of $^*D$ vanishes then M is of constant φ-holomorphic sectional curvature.

For this aim, by the quite similar process to chapter I, §2, we construct, in the sequel, a φ-connection $^*D$ whose components $^*\Gamma_{ji}^{h}$ are given by

(2.1) $$^*\Gamma_{ji}^{h}=\Gamma_{ji}^{h}-p_j\varphi_i^{h}-\varphi_j^{h}q_i-\gamma_j^{h}p_i,$$

where Γ_{ji}^{h} are defined by (1.5), that is,

$$(2.2) \qquad {}^*\Gamma_{ji}{}^h = \left\{ {}_j{}^h{}_i \right\} + q_j\varphi_i{}^h - p_j\varphi_i{}^h - \gamma_{ji}p^h - \varphi_{ji}q^h.$$

Denoting by ${}^*D_j$ the operator of covariant differentiation with respect to ${}^*\Gamma_{ji}{}^h$, we find

$$(2.3) \qquad {}^*D_k g_{ji} = \gamma_{kj}p_i + \gamma_{ki}p_j + \varphi_{kj}q_i + \varphi_{ki}q_j,$$

$$(2.4) \qquad {}^*D_k\varphi_{ji} = \gamma_{kj}q_i - \gamma_{ki}q_j - \varphi_{kj}p_i + \varphi_{ki}p_j. \quad (\text{or } {}^*D_k\varphi_j{}^h = 0).$$

In this case, we obtain

$$(2.5) \qquad {}^*\Gamma_{ji}{}^h - {}^*\Gamma_{ij}{}^h = -2\varphi_{ji}q^h + (q_j - p_j)\varphi_i{}^h - (q_i - p_i)\varphi_j{}^h,$$

and we call ${}^*D$ a *φ-connection which is closely related to the cosymplectic Weyl–Hlavaty connection.*

Computing the curvature tensor ${}^*R_{kji}{}^h$ of ${}^*\Gamma_{ji}{}^h$ and using the notations (1.8), (1,9) and

$$p_k{}^h = p_{kt}g^{th}, \quad q_k{}^h = q_{kt}g^{th},$$

we obtain

$$(2.6) \qquad \begin{aligned} {}^*R_{kji}{}^h = K_{kji}{}^h &- \gamma_{ji}(p_k{}^h - \frac{\lambda}{2}\gamma_k{}^h) + \gamma_{ki}(p_j{}^h - \frac{\lambda}{2}\gamma_j{}^h) \\ &- \varphi_{ji}(q_k{}^h - \frac{\lambda}{2}\varphi_k{}^h) + \varphi_{ki}(q_j{}^h - \frac{\lambda}{2}\varphi_j{}^h) \\ &+ (q_{kj} - q_{jk} - \lambda\varphi_{kj})\varphi_i{}^h - (p_{kj} - p_{jk})\varphi_i{}^h. \end{aligned}$$

§ 3. *A characterization of a cosymplectic manifold by the curvature tensor ${}^*R_{kji}{}^h$ defined by* (2.6).

In this section, we prove the following

THEOREM 3.1 *Let a real $(2n+1)$-dimensional cosymplectic manifold M $(n \geqq 2)$ admits the cosymplectic Weyl-Hlavaty connection. If the curvature tensor formed with the components of the connection ${}^*D$ defined by* (2.2) *vanishes locally, then M is of constant φ-holomorphic sectional curvature locally, and vice-versa.*

Proof. Assuming that

$$(3.1) \qquad {}^*R_{kji}{}^h = 0,$$

we can write (2.6) in the following covariant form:

$$(3.2) \qquad \begin{aligned} K_{kjih} = \gamma_{ji}(p_{kh} - \frac{\lambda}{2}\gamma_{kh}) &- \gamma_{ki}(p_{jh} - \frac{\lambda}{2}\gamma_{jh}) \\ + \varphi_{ji}(q_{kh} - \frac{\lambda}{2}\varphi_{kh}) &- \varphi_{ki}(q_{jh} - \frac{\lambda}{2}\varphi_{jh}) \end{aligned}$$

$$-(q_{kj}-q_{jk}-\lambda\varphi_{kj})\varphi_{ih}-(p_{kj}-p_{jk})\varphi_{ih}.$$

For briefness, we now put

$$P_{ji}=p_{ji}-\frac{\lambda}{2}\gamma_{ji}, \tag{3.3}$$

$$Q_{ji}=q_{ji}-\frac{\lambda}{2}\varphi_{ji}. \tag{3.4}$$

Then (3.2) becomes

$$\begin{aligned} K_{kjih}=&\gamma_{ji}P_{kh}-\gamma_{ki}P_{jh}+\varphi_{ji}Q_{kh}-\varphi_{ki}Q_{jh} \\ &-(Q_{kj}-Q_{jk})\varphi_{ih}+(P_{kj}-P_{jk})\varphi_{ih}. \end{aligned} \tag{3.5}$$

Transvecting (3.3) or (3.4) with $\varphi_k{}^i$ and using (1.12), we find

$$P_{jt}\varphi_k{}^t=-Q_{jk}, \qquad Q_{jt}\varphi_k{}^t=P_{jk}. \tag{3.6}$$

From (3.6), we obtain

$$P_{jt}\xi^t=0, \qquad Q_{jt}\xi^t=0. \tag{3.7}$$

Transvecting (3.5) with g^{kh} and taking account of (3.6) and (3.7), we obtain

$$K_{ji}=P\gamma_{ji}+Q\varphi_{ji}+P_{ji}+Q_{ji}-Q_{tj}\varphi_i{}^t+P_{tj}\varphi_i{}^t, \tag{3.8}$$

where we have put $P=P_t{}^t$ and $Q=Q_t{}^t$.

Transvecting (3.8) with $\varphi_k{}^i$ and taking account of (3.6), we find

$$K_{jt}\varphi_i{}^t=P\varphi_{ji}+Q\gamma_{ji}+P_{ji}-P_{ij}-Q_{ji}+Q_{ij}. \tag{3.9}$$

Substituting (3.9) into the well known equation

$$K_{jt}\varphi_i{}^t+K_{it}\varphi_j{}^t=0,$$

we obtain

$$Q=0. \tag{3.10}$$

Substituting (3.5) into the well known identity

$$K_{kjih}-K_{ihkj}=0 \tag{3.11}$$

and transvecting it with γ^{ih}, we obtain

$$P_{jk}=P_{kj} \tag{3.12}$$

by virtue of (3.6) and (3.10).

Substituting (3.5) into (3.11) and transvecting it with φ^{kh}, we find

$$P\varphi_{ji}+(2n+1)Q_{ij}-(Q_{tj}\gamma_i{}^t+Q_{ti}\gamma_j{}^t)-Q_{tj}\gamma_i{}^t=0. \tag{3.13}$$

On the other hand, taking account of (3.6), (3.7) and (3.12), we find

(3.14) $$P_{tj}\xi^t=0,\quad Q_{tj}\xi^t=0.$$

Then, (3.13) becomes

(3.15) $$P\varphi_{ji}+(2n-1)Q_{ij}-Q_{ji}=0.$$

Taking the skew-symmetric part of (3.15), we find

(3.16) $$Q_{ji}+Q_{ij}=0.$$

Substituting (3.16) into (3.15), we obtain

(3.17) $$Q_{ji}=\frac{P}{2n}\varphi_{ji}.$$

Substituting (3.17) into (3.6), we obtain

(3.18) $$P_{ji}=\frac{P}{2n}\gamma_{ji}.$$

Substituting (3.10), (3.17) and (3.18) into (3.8), we find

(3.19) $$K_{ji}=(1+\frac{1}{n})P\gamma_{ji},$$

from which, we obtain

(3.20) $$K=2(n+1)P.$$

Substituting (3.19) and (3.20) into the identity $\nabla_j K=2\nabla_t K_j{}^t$, we obtain $(1-n)\nabla_j P=\xi^t(\nabla_t P)\eta_j$. Transvecting it with ξ^j, we find $\xi^t\nabla_t P=0$, *by* $n\geqq 2$, from which $\nabla_j P=0$, that is,

$$P=\text{const.}=\frac{K}{2(n+1)}.$$

Substituting (3.17) and (3.18) into (3.5) and taking account of (3.20), we obtain

(3.21) $$K_{kji}{}^h=\frac{K}{4n(n+1)}[\gamma_{ji}\gamma_k{}^h-\gamma_{ki}\gamma_j{}^h+\varphi_{ji}\varphi_k{}^h-\varphi_{ki}\varphi_j{}^h-2\varphi_{kj}\varphi_i{}^h],$$

K being a constant.

Therefore M is of constant φ-holomorphic sectional curvature. (cf. [2]).

Conversely, if M is of constant φ-holomorphic sectional curvature, then the Riemann-Christoffel curvature tensor of M is of the form (3.21). In this case, we consider the integrability condition of the differential equations

(3.22) $$p_i=\nabla_i p$$

$$\nabla_j p_i = p_j p_i - q_j q_i + \frac{K}{4n(n+1)} \gamma_{ji},$$

where K is a constant and

(3. 23) $$q_j = -p_t \varphi_j{}^t.$$

By a straightforward computation, we see that the following equation is satisfied

$$\nabla_k \nabla_j p_i - \nabla_j \nabla_k p_i = -K_{kji}{}^t p_t$$

by the help of (1. 6) and (3. 21).

Therefore a gradient covector field p_i satisfying (3. 22) is completely integrable locally.

In this case, differentiating covariantly (3. 23) and substituting (3. 22) into it, we easily obtain

(3. 24) $$\nabla_j q_i = p_j q_i + q_j p_i + \frac{K}{4n(n+1)} \varphi_{ji}.$$

Substituting (3. 22) and (3. 24) into (1. 8) and (1. 9) respectively, we obtain

(3. 25) $$p_{ji} - \frac{\lambda}{2} \gamma_{ji} = \frac{1}{4n(n+1)} \gamma_{ji},$$

(3. 26) $$q_{ji} - \frac{\lambda}{2} \varphi_{ji} = \frac{1}{4n(n+1)} \varphi_{ji}.$$

Substituting (3. 25) and (3. 26) into (2. 6), we see that

$$*R_{kji}{}^h = 0.$$

Thus, the theorem 3. 1 is completely proved.

III. A characterization of a Sasakian manifold by a curvature tensor formed with an η-connection with torsion.

§ 1. *Preliminaries.*

We consider a Sasakian manifold M of real $2n+1$ dimensions covered by a coordinate neighborhoods $\{U; y^h\}$ and denote by g_{ji}, $\varphi_j{}^h$, ξ^h and η_j components of the Grayan metric tensor, those of the Sasakian structure tensor, those of the Sasakian vector and those of the Sasakian 1-form of M respectively, where, here and in the sequel, the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$.

We denote by $K_{kji}{}^h$, K_{ji} and K the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

In a previous paper [1], we considered an affine connection $'D$ satisfying (1.1)-(1.4) of chapter II in a Sasakian manifold M and we called it a *contact Weyl–Hlavaty connection.*

Furthermore in [1], we also considered an affine connection D whose components Γ_{ji}^h are related to the components $'\Gamma_{ji}^h$ of $'D$ by $\Gamma_{ji}^h = {}'\Gamma_{ji}^h - p_j \gamma_i^h$, where $\gamma_{ji} = g_{ji} - \eta_j \eta_i$, that is,

$$(1.1) \qquad \Gamma_{ji}^h = \left\{ {h \atop j\ i} \right\} + \gamma_j^h p_i - \gamma_{ji} p^h + \varphi_j^h (q_i - \eta_i) + \varphi_i^h (q_j - \eta_j) - \varphi_{ji} (q^h - \xi^h),$$

for a certain non-zero covector p_i, where $p^h = p_t g^{th}$,

$$(1.2) \qquad q_i = -p_t \varphi_i^t, \quad p_i = q_t \varphi_i^t, \quad \gamma_j^h = \gamma_{jt} g^{th}, \quad q^h = q_t g^{th}$$

in a Sasakian manifold M.

(Such a connection D is called "a special semi-symmetric metric φ-connection" in [5].)

Denoting by R_{kji}^h the curvature tensor of Γ_{ji}^h, we obtain (cf. [1])

$$(1.3) \qquad \begin{aligned} R_{kji}^h = K_{kji}^h &- \gamma_k^h p_{ji} + \gamma_j^h p_{ki} - p_k^h \gamma_{ji} + p_j^h \gamma_{ki} \\ &- \varphi_k^h q_{ji} + \varphi_j^h q_{ki} - q_k^h \varphi_{ji} + q_j^h \varphi_{ki} \\ &- \alpha_{kj} \varphi_i^h - \varphi_{kj} \beta_i^h + (\varphi_k^h \varphi_{ji} - \varphi_j^h \varphi_{ki} - 2\varphi_{kj} \varphi_i^h), \end{aligned}$$

where

$$(1.4) \qquad p_{ji} = \nabla_j p_i - p_j p_i + (q_j - \eta_j)(q_i - \eta_i) + \frac{1}{2} \lambda \gamma_{ji},$$

$$(1.5) \qquad q_{ji} = \nabla_j q_i - p_j (q_i - \eta_i) - p_i (q_j - \eta_j) + \frac{1}{2} \lambda \varphi_{ji},$$

λ being defined by $\lambda = p_t p^t = q_t q^t$,

$$(1.6) \qquad \alpha_{ji} = -(\nabla_j q_i - \nabla_i q_j),$$

$$(1.7) \qquad \beta_{ji} = 2(p_j q_i - p_i q_j)$$

and $p_k^h = p_{kt} g^{th}$, $q_k^h = q_{kt} g^{th}$, $\beta_k^h = \beta_{kt} g^{th}$.

The following relations are easily checked. (cf. [1])

$$(1.8) \qquad p_{jt} \xi^t = \eta_j, \quad q_{jt} \xi^t = 0.$$

$$(1.9) \qquad p_{jt} \varphi_i^t = -q_{ji}, \quad q_{jt} \varphi_i^t = p_{ji} - \eta_j \eta_i.$$

$$(1.10) \qquad \alpha_{ji} = -(q_{ji} - q_{ij} - \lambda \varphi_{ji}).$$

$$(1.11) \qquad \alpha_{jt} \xi^t = q_{tj} \xi^t, \quad \beta_{jt} \xi^t = 0.$$

§2. *A certain η-connection which is closely related to the contact Weyl-Hlavaty connection.*

In this section, we want to seek out an η-connection $^*D$ in a Sasakian manifold M such that if the curvature tensor formed with the components of $^*D$ vanishes then M is of constant C-holomorphic sectional curvature.

For this aim, by the quite similar process to chapter I, §2, we construct, in the sequel, an η-connection $^*D$ whose connection $^*\Gamma_{ji}^h$ are given by

$$(2.1) \qquad {}^*\Gamma_{ji}^h=\Gamma_{ji}^h-p_j\varphi_i^h-\varphi_j^h q_i-\gamma_j^h p_i,$$

where Γ_{ji}^h are defined by (1.1), that is,

$$(2.2) \qquad {}^*\Gamma_{ji}^h=\left\{{}_j{}^h{}_i\right\}+\gamma_i^h p_j-\gamma_{ji}p^h-\varphi_i^h p_j-\varphi_j^h\eta_i +\varphi_i^h(q_j-\eta_j)-\varphi_{ji}(q^h-\xi^h).$$

Denoting by $^*D_j$ the operator of covariant differentiation with respect to $^*\Gamma_{ji}^h$, we find

$$(2.3) \qquad {}^*D_k g_{ji}=\gamma_{kj}p_i+\gamma_{ki}p_j+\varphi_{kj}q_i+\varphi_{ki}q_j,$$

$$(2.4) \qquad {}^*D_k\varphi_{ji}=\gamma_{kj}q_i-\gamma_{ki}q_j-\varphi_{kj}p_i+\varphi_{ki}p_j,$$

$$(2.5) \qquad {}^*D_k\eta_i=0, \quad {}^*D_k\gamma_j^h=0.$$

In this case we obtain

$$(2.6) \qquad {}^*\Gamma_{ji}^h-{}^*\Gamma_{ij}^h=\gamma_i^h p_j-\gamma_j^h p_i-\varphi_i^h p_j+\varphi_j^h p_i +\varphi_i^h q_j-\varphi_j^h q_i-2\varphi_{ji}(q^h-\xi^h),$$

and we call $^*D$ an *η-connection which is closely related to the contact Weyl-Hlavaty connection.*

Computing the curvature tensor $^*R_{kji}^h$ of $^*\Gamma_{ji}^h$ and using the notations (1.4), (1.5) and

$$p_k^h=p_{kt}g^{th}, \qquad q_k^h=q_{kt}g^{th},$$

we obtain

$$(2.7) \qquad \begin{aligned} {}^*R_{kji}^h=&K_{kji}^h-\gamma_{ji}\left(p_k^h-\frac{\lambda}{2}\gamma_k^h\right)+\gamma_{ki}\left(p_j^h-\frac{\lambda}{2}\gamma_j^h\right)\\ &-\varphi_{ji}\left[q_k^h-\left(1+\frac{\lambda}{2}\right)\varphi_k^h\right]+\varphi_{ki}\left[q_j^h-\left(1+\frac{\lambda}{2}\right)\varphi_j^h\right]\\ &+\left[q_{kj}-q_{jk}-(2+\lambda)\varphi_{kj}\right]\varphi_i^h+(\gamma_j^h\eta_k-\gamma_k^h\eta_j)\eta_i\\ &-(p_{kj}-p_{jk})\varphi_i^h. \end{aligned}$$

§3. *A characterization of a Sasakian manifold by the curvature tensor* $^*R_{kji}{}^h$ *defined by* (2.7).

In this section we prove the following

THEOREM 3.1. *Let a real* $(2n+1)$*-dimensional Sasakian manifold* M $(n\geqq 2)$ *admits the contact Weyl-Hlavaty connection. If the curvature tensor formed with the components of the connection* $^*D$ *defined by* (2.2) *vanishes locally, then* M *is of constant C-holomorphic sectional curvature locally, and vice-versa.*

Proof. Assuming that

$$^*R_{kji}{}^h=0, \tag{3.1}$$

we have from (2.7)

$$\begin{aligned}K_{kji}{}^h=&\gamma_{ji}P_k{}^h-\gamma_{ki}P_j{}^h+\varphi_{ji}Q_k{}^h-\varphi_{ki}Q_j{}^h-(Q_{kj}-Q_{jk})\varphi_i{}^h\\&+(P_{kj}-P_{jk})\varphi_i{}^h-(\gamma_j{}^h\eta_k-\gamma_k{}^h\eta_j)\eta_i,\end{aligned} \tag{3.2}$$

where we have put

$$P_{kj}=p_{kj}-\frac{\lambda}{2}\gamma_{kj},\qquad P_k{}^i=P_{kj}g^{ji} \tag{3.3}$$

and

$$Q_{kj}=q_{kj}-\left(1+\frac{\lambda}{2}\right)\varphi_{kj},\qquad Q_k{}^i=Q_{kj}g^{ji}. \tag{3.4}$$

Transvecting (3.3) with $\varphi_i{}^h$ and using (1.9), we obtain

$$P_{kt}\varphi_j{}^t=-Q_{kj}-\varphi_{kj}. \tag{3.5}$$

Similarly from (1.9) and (3.4), we obtain

$$Q_{kt}\varphi_j{}^t=P_{kj}-g_{kj}. \tag{3.6}$$

Transvecting (3.3) with ξ^j and using (1.8), we obtain

$$P_{jt}\xi^t=\eta_j. \tag{3.7}$$

Similarly from (1.8) and (3.4), we obtain

$$Q_{jt}\xi^t=0. \tag{3.8}$$

Contracting with respect to k and h in (3.2) and taking account of (3.5) and (3.6), wd obtain

$$K_{ji}=(P-2)\gamma_{ji}+(Q+1)\varphi_{ji}+P_{ji}+Q_{ji}+P_{tj}\varphi_i{}^t-Q_{tj}\varphi_i{}^t+(2n-1)\eta_j\eta_i, \tag{3.9}$$

where we have put

$$P=P_t{}^t,\qquad Q=Q_t{}^t. \tag{3.10}$$

Transvecting (3. 9) with $\varphi_k{}^i$ and taking account of (3. 5) and (3. 6), we obtain

$$(3.11)\qquad \begin{aligned} K_{jt}\varphi_i{}^t = &-(P-1)\varphi_{ji}+(Q+1)\gamma_{ji}-Q_{ji}+P_{ji}-g_{ji} \\ &-(P_{ij}-\eta_i P_{tj}\xi^t)+(Q_{ij}-\eta_i Q_{tj}\xi^t). \end{aligned}$$

Substituting (3. 11) into the well known equation

$$K_{jt}\varphi_i{}^t+K_{it}\varphi_j{}^t=0,$$

we have

$$(3.12)\qquad 2(Q+1)\gamma_{ji}-2g_{ji}+(\eta_i P_{tj}+\eta_j P_{ti})\xi^t-(\eta_i Q_{tj}+\eta_j Q_{ti})\xi^t=0.$$

Transvecting (3. 12) with g^{ji}, we see that

$$(3.13)\qquad Q=0.$$

Substituting (3. 13) into (3. 12) and transvecting it with ξ^i, we obtain

$$(3.14)\qquad P_{tj}\xi^t-Q_{tj}\xi^t=\eta_j.$$

Substituting (3. 2) into the well known identity

$$(3.15)\qquad K_{kjih}-K_{ihkj}=0$$

and transvecting it with γ^{ih}, we obtain

$$(3.16)\qquad P_{kj}=P_{jk}$$

by the help of (3. 5), (3. 6) and (3. 13).

Taking account of (3. 7), (3. 14) and (3. 16), we see that

$$(3.17)\qquad Q_{tj}\xi^t=0.$$

Substituting (3. 2) into (3. 15) and transvecting it with φ^{kh}, we obtain

$$(3.18)\qquad (P-2n-1)\varphi_{ji}=Q_{ji}+(1-2n)Q_{ij}$$

by the help of (3. 5), (3. 6) and (3. 16).

Since $\varphi_{ji}+\varphi_{ij}=0$, we see from (3. 18) that

$$(3.19)\qquad Q_{kj}+Q_{jk}=0.$$

Therefore (3. 18) is rewritten as

$$(3.20)\qquad Q_{ji}=\left(\frac{P-1}{2n}-1\right)\varphi_{ji}.$$

Substituting (3. 20) into (3. 16), we obtain

$$(3.21)\qquad P_{ji}=\frac{P-1}{2n}g_{ji}-\left(\frac{P-1}{2n}-1\right)\eta_j\eta_i.$$

On the other hand, if we take account of (3.5), (3.6), (3.13), (3.16) and (3.19), then (3.9) is rewritten as

$$K_{ji}=(P-1)g_{ji}+(2n+1-P)\eta_j\eta_i. \tag{3.22}$$

Transvecting (3.22) with g^{ji} we easily see that

$$K=2nP. \tag{3.23}$$

Substituting (3.22) and (3.23) into the well known identity $\nabla_j K=2\nabla_t K_j{}^t$, we see that $(1-n)\nabla_j P=\xi^t(\nabla_t P)\eta_j$. Transvecting it with ξ^j, we find $\xi^t\nabla_t P=0$ by the help of $n\geqq 2$, from which $\nabla_j P=0$, that is, $P=$const. Thus M is an η-Einstein manifold ([3]).

Now we put

$$\frac{P-1}{2n}-1=k \tag{3.24}$$

k being a constant.

Then, (3.20) and (3.21) are respectively rewritten as

$$Q_{ji}=k\varphi_{ji}. \tag{3.25}$$

$$P_{ji}=k\gamma_{ji}+g_{ji}. \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.2) and recalling $\gamma_{ji}=g_{ji}-\eta_j\eta_i$, we obtain

$$\begin{aligned}K_{kjih}=(k+1)(g_{kh}g_{ji}-g_{jh}g_{ki})-k[\eta_k(g_{ji}\eta_h-g_{jh}\eta_i)\\ -\eta_j(g_{ki}\eta_h-g_{kh}\eta_i)-\varphi_{ji}\varphi_{kh}+\varphi_{ki}\varphi_{jh}+2\varphi_{kj}\varphi_{ih}].\end{aligned} \tag{3.27}$$

Therefore M is of a constant C-holomorphic sectional curvature ([3]).

Conversely, let the Riemann-Christoffel curvature tensor of M be of the form (3.27). We consider the integrability condition of the differential equations

$$\begin{aligned}&\nabla_i p=p_i.\\ &\nabla_j p_i=(k+1)g_{ji}+p_jp_i-q_jq_i+q_j\eta_i+q_i\eta_j-(k+1)\eta_j\eta_i,\end{aligned} \tag{3.28}$$

where

$$q_j=-p_t\varphi_j{}^t. \tag{3.29}$$

By a straightforward computation, we see that the following equation is satisfied

$$\nabla_k\nabla_j p_i-\nabla_j\nabla_k p_i=-K_{kji}{}^t p_t$$

by the help of (1.2) and (3.27).

Therefore a gradient covector field p_i satisfying (3.28) is completely integrable locally.

In this case, differentiating covariantly (3.29) and substituting (3.28) into it, we easily obtain

$$\nabla_j q_i = p_j q_i + q_j p_i - \eta_j p_i - p_j \eta_i + (k+1)\varphi_{ji}. \tag{3.30}$$

Substituting (3.28) and (3.30) into (1.4) and (1.5) respectively, we obtain

$$p_{ji} - \frac{\lambda}{2}\gamma_{ji} = (k+1) g_{ji} - k\eta_j \eta_i, \tag{3.31}$$

$$q_{ji} - \left(1+\frac{\lambda}{2}\right)\varphi_{ji} = k\varphi_{ji}. \tag{3.32}$$

Substituting (3.31) and (3.32) into (2.7), we see that

$$^*R_{kji}{}^h = 0.$$

Thus, the theorem is completely proved.

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