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REGULAR HOMOMORPHISMS

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In this paper, we will show that an epimorphism $\phi:(X,T)\to(Y,T)$ is distal and regular if and only if there exists a group H of automorphisms of (X,T) such that (H,X,T) is a bitransformation group and ϕ induces an isomorphism of $(X/H,T)$ onto (Y,T), where (Y,T) is pointwise almost periodic.

In this paper, let T be an arbitrary, but fixed, topological group and we consider the (right) transformation group (X,T) with a compact Hausdorff phase space X. A closed nonempty subset A of X is said to be a *minimal set* if, for every $x\in A$, the orbit xT is a dence subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point.* If X is itself minimal, we say it is a *minimal transformation group.* The points x and y of X in the transformation group (X,T) are called *proximal* provided that for each neighborhood W of the diagonal Δ of $X\times X$, there exists a $t\in T$ such that $(xt,yt)\in W$. We denote $P(X,T)=\{(x,y)\in X\times X\mid x$ and y are proximal$\}$ which is called the *proximal relation* on (X,T). The transformation group (X,T) is said to be *proximal* if every two points of X are proximal.

If (Y,T) is also a transformation group, a *homomorphism* from (X,T) to (Y,T) is a continuous map $\phi:X\to Y$ such that $\phi(xt)=\phi(x)t$ $(x\in X,\ t\in T)$. Especially, if ϕ is bijective from X onto X itself, then ϕ is called an *automorphism* of (X,T).

In [3], Ellis showed that if (X,T) is a transformation group and $E(X)$ its enveloping semigroup, then for each $x\in X$ the map $\theta_x:p\to xp=p(x)$ of $E(X)$ into X is a homomorphism, and its image is just the orbit closure $\overline{xT}$ of x.

The homomorphism $\phi:(X,T)\to(Y,T)$ is said to be *distal* provided $\phi(x)=\phi(y)$ and $x\neq y$ imply $(x,y)\notin P(X,T)$. The homomorphism $\phi:(X,T)\to(Y,T)$ is said to be *proximal* if for each $x,x'\in\phi^{-1}(y)$, $(x,x')\in P(X,T)$ $(y\in Y)$.

We define a new homomorphism for the main Theorem as follows:

DEFINITION 1. The homomorphism $\phi:(X,T)\to(Y,T)$ is said to be *reg-*

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ular if for each $x, x' \in \phi^{-1}(y)$, there exists an automorphism h of (X, T) such that $(hx, x') \in P(X, T)$ and $\phi h = \phi$ $(y \in Y)$.

REMARK 2. (1) Let $\phi : (X, T) \to (Y, T)$ be a proximal homomorphism, then ϕ is regular (take h to be the identity). In general, the converse does not hold.

(2) If (X, T) and (Y, T) are minimal transformation groups then a homomorphism $\phi : (X, T) \to (Y, T)$ is regular if and only if for each $x, x' \in \phi^{-1}(y)$, there exists an automorphism h of (X, T) such that $(hx, x') \in P(X, T)$.

(3) Let $\phi : (X, T) \to (Y, T)$ be a homomorphism and (X, T) be a regular minimal set, then ϕ is a regular homomorphism.

Proof: We need only prove (2). Let $\phi(x) = \phi(x')$. There is an automorphism h of (X, T) such that $(hx, x') \in P(X, T)$. Therefore there is a minimal right ideal I of $E(X)$ such that $(hx)p = x'p$ $(p \in I)$. Now we have

$$\begin{aligned} \phi h(xp) &= \phi(h(x)p) = \phi(x'p) = \phi(x')\theta(p) \\ &= \phi(x)\theta(p) = \phi(xp). \end{aligned}$$

Thus ϕh and ϕ agree at a point of (X, T). Therefore we have $\phi h = \phi$.

REMARK 3. Ellis showed in Remark 4.6 in [3] that if (X, T) is uniformly almost periodic and minimal; $x \in X$, and $\theta_x : (E(X), T) \to (Y, T)$ is defined by $\theta_x(p) = xp$ $(p \in E(X))$, then there is a topological group H such that $(H, E(X), T)$ is a bitransformation group and θ_x induces an isomorphism of $(E(X)/H, T)$ onto (X, T).

We show in fact that this homomorphism θ_x is distal and regular.

(1) Since (X, T) is uniformly almost periodic, (X, T) is distal. Therefore $(E(X), T)$ is also distal. Hence θ_x is distal.

(2) Since $(E(X), T)$ is a minimal right ideal, $\theta_x : (E(X), T) \to (X, T)$ is a regular homomorphism.

The result of Remark 3 is no accident, as shown by the following:

THEOREM 4. *Let $\phi : (X, T) \to (Y, T)$ be an epimorphism and (Y, T) be pointwise almost periodic. Then ϕ is distal and regular if and only if there exists a group H of automorphisms of (X, T) such that (H, X, T) is a bitransformation group and ϕ induces an isomorphism of $(X/H, T)$ onto (Y, T).*

Proof. If: (1) We show that ϕ is distal. Let $\phi(x) = \phi(y)$. Since $(X/H, T)$ is pointwise almost periodic, $\phi(x)$ is an almost periodic point of $(X/H, T)$. By Proposition 6.1 in [3], there exists an almost periodic point z of X such that $\phi(z) = \phi(x)$. Thus x and y belong to the same class of z (that is, $x, y \in Hz$). Hence we let $x = h'z$ and $y = hz$.

Suppose $(x, y)\in P(X, T)$, then there exists a net (t_α) such that $\lim(h'z)t_\alpha=\lim(hz)t$.

Since h and h' are automorphisms from X to X and we may assume $\lim zt_\alpha$ exists, we have

$$h(\lim zt_\alpha)=h'(\lim zt_\alpha).$$

If we let $w=\lim zt_\alpha$, then $hw=h'w$ and $w\in\overline{zT}$. Since z is an almost periodic point of X, $\overline{zT}$ is minimal. Thus we obtain $y=hz=h'z=x$ since h and h' agree at a point w in $\overline{zT}$. Therefore ϕ is distal.

(2) We show that ϕ is regular. Let $x, x'\in\phi^{-1}(y)=Hz$. Then x' belong to $Hx(=Hz)$. Therefore there exists an element $h\in H$ such that $hx=x'$. Since $h\in H$, h is an automorphism of (X, T) and $(hx, x')\in P(X, T)$.

Let x'' be any element of (X, T). Then $\phi hx''=\phi(hx'')=H(hx'')=Hx''=\phi x''$. Thus ϕ is a regular homomorphism.

Only if: Let $\phi:(X, T)\to(Y, T)$ be a distal and regular epimorphism and G be the group of automorphisms of (X, T). Define

$$H=\{h\mid\phi h=\phi,\ h\in G\}.$$

Then H is a group of automorphisms of (X, T). Now, it is trivial that (H, X, T) is a bitransformation group. Therefore it is sufficient to show that ϕ induces an isomorphism of $(X/H, T)$ onto (Y, T).

Let $K=\{(x, x')\mid\phi(x)=\phi(x')\}$. Since (X, T) is compact Hausdorff and ϕ is a homomorphism from (X, T) onto (Y, T), $(X/K, T)$ is isomorphic to (Y, T).

Since $(X/K, T)$ and $(X/H, T)$ have the identification topology induced by X and the acting group is also induced from (X, T), it is sufficient to show that $Kx=Hx$ for each $x\in X$. For each $h\in H$, $\phi(hx)=\phi h(x)=\phi(x)$. Therefore hx belongs to Kx. Conversely, for each $x'\in Kx$, there exists an automorphism h such that $(hx, x')\in P(X, T)$ and $\phi h=\phi$ since ϕ is regular. Thus $h\in H$. Since ϕ is distal and $\phi(hx)=\phi(x)=\phi(x')$, we have $x'=hx$. Thus we obtain $Kx=Hx$. Therefore $(X/H, T)$ is isomorphic to (Y, T).

In [4], Memahon showed that given a distal homomorphism ϕ from a minimal set (X, T) onto a minimal set (Y, T), there exists a bitransformation group (K, W, T), and a subgroup L of K such that (W, T) is minimal, K acts freely on W, and $(W/L, T)$ can be identified with (X, T), $(W/K, T)$ can be identified with (Y, T) and the natural projection of W/L onto W/K can be identified with ϕ.

Proposition 6.6 of [3] is a corollary of Theorem 4 as follows:

COROLLARY 5: *Let (H, X, T) be a bitransformation group such that X/H is Hausdorff and $(X/H, T)$ is distal. Then (X, T) is distal.*

REMARK 6. (1) In Theorem 4, the distality of alone does not imply the right side of Theorem 4; that is, there exists a topological group H such that (H, X, T) is a bitransformation group and ϕ induces an isomorphism of $(X/H, T)$ onto (Y, T). Because, if we consider a minimal distal transformation group (X, T) such that (X, T) is not regular minimal, (we will give an example of such a transformation group), and $\phi : (X, T) \to (Y, T)$ is a homomorphism onto a one point space (Y, T), then ϕ is distal. Suppose there exists a topological group H such that (H, X, T) is a bitransformation group and ϕ induces an isomorphism of $(X/H, T)$ onto (Y, T). By Theorem 4, ϕ is also regular. Therefore (X, T) is regular minimal. This contradicts our hypothesis on (X, T).

(2) The regularity of ϕ alone does not imply the right side of Theorem 4. Because, let (X, T) be a nontrivial proximal transformation group and $\phi : (X, T) \to (Y, T)$ be a homomorphism, where (Y, T) is a one point space. Then ϕ is proximal. Therefore ϕ is also regular by Remark 2. If the right side of Theorem 4 holds, then ϕ is also distal by Theorem 4. Since the one point space (Y, T) is distal and ϕ is distal, (X, T) is also distal. This contradicts the assumption that (X, T) is a nontrivial proximal transformation group.

EXAMPLE 7. Let $X = \{1, 2, 3\}$ be a discrete topological space and $T = S_3$ be the permutation group of X. Then (X, T) is a minimal distal transformation group. But (X, T) is not regular minimal since (X, T) is not isomorphic to $(E(X), T)$.

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