J. Korean Math, Soc.
Vol. 18, No. 2, 1982

THE JACOBSON DENSITY THEOREM AND SOME APPLICATIONS

BY B. S. CHWE AND J. NEGGERS

1. The density theorem

Consider the ring $\Omega=\mathrm{Hom}(M, M)$, where M is an abelian group. Let R and S be rings contained in Ω, such that ${}_RM$ is an irreducible R-module and such that $S=\{\alpha\in\Omega \mid \alpha r=r\alpha$ for all $r\in R\}$. Then S is a division ring and if M_S is a finite dimensional vector space, R is isomorphic to a ring of matrices over S.

This result can be generalized as in the following two theorems. Furthermore it will be clear that if R is a primitive ring and if ${}_RM$ is a faithful simple R-module, then R and ${}_RM$ satisfy the conditions of theorems 1 and 2, so that the usual Jacobson density theorem is a consequence of theorem 2.

THEOREM 1. *Let R be a ring, ${}_RM$ a faithful R-module, $S=\mathrm{Hom}_R({}_RM, {}_RM)$ and M_S the corresponding S-module. Suppose that if ${}_RN$ is a proper submodule of ${}_RM$, then there is a nonzero element f of S such that $Nf=0$. Also, suppose that if $a\in M$, $Ra=0$, then $a=0$.*

If V is a finite dimensional free S-submodule of M, if $a\in M$ and if $A(V)=\{x\in R \mid xV=0\}$, then $A(V)a=0$ implies $a\in V$. If a basis of V together with a forms an S-linearly independent set, then $A(V)a=M$.

THEOREM 2. *Let $R, S, {}_RM$ and M_S be as in* theorem 1 *and suppose that the hypotheses of* theorem 1 *hold. Let M_S be a free S-module. Then there is a monomorphism ξ from R to $R^*=\mathrm{Hom}_S(M_S, M_S)$ such that $\xi(R)$ is dense in R^*.*

Proof of theorem 2. Identify R as a subring of R^*. Let $U=\{u_1, \cdots, u_m\}$ be a linearly independent subset of M over S and let $\{v_1, \cdots, v_m\}$ be a subset of M. If W_j is the s-span of $\{u_1, \cdots, \widehat{u_j}, \cdots, u_m\}$, then $A(W_j)u_j=M$, and hence there is an element r_j of $A(W_j)$ such that $r_ju_j=v_j$ and $r_ju_m=0$ if $i\neq j$. If $r=\sum_i r_i$, then $ru_j=\sum_i r_iu_j=r_ju_j=v_j$ for all j. Hence R is dense in R^*.

Proof of theorem 1. Suppose $V=\{0\}$, Then $A(V)=R$, so that $Ra=0$ implies $a=0$. Also, if $\{a\}$ is linearly independent over S, then $Ra=M$, since

Received Sep. 5, 1981.

otherwise $(Ra)f=0$, whence $(a)f=af=0$, contradicting the fact that $\{a\}$ is linearly independent over S. Now, if $V=W+uS$, with u S-linearly independent of (a basis of) W, and if the conclusions of the theorem hold for W, then $A(W)u=M$. Suppose this is so and that $A(V)a=0$. If $x\in A(W)$ and $xu=0$, then $x\in A(V)$, so that also $xa=0$. Thus, there is an R-homomorphism θ from $A(W)u$ to $A(W)a$ given by $\theta: xu\rightarrow xa$ for all $x\in A(W)$. Since $A(W)\cdot(u\theta-a)=0$, $u\theta-a=-w\in W$, so that $w+u\theta=a\in V$. The first conclusion follows.

Next, suppose that a basis of V together with a forms an S-linearly independent set. Also suppose that $A(V)a\neq M$. Hence there is a nonzero element f of S such that $(A(V)a)f=A(V)(af)=0$. Since $f\neq 0$ and since $\{a\}$ is S-linearly independent $af\neq 0$.

Consider the map θ from $A(W)u$ to $A(W)(af)$ given by $\theta: xu\rightarrow x(af)$. Note that $A(W)u=M$ by the induction hypothesis. Let $x\in A(W)$; then if $xu=0$, $x\in A(V)$ and $x(af)=0$. Hence θ is a well-defined R-endomorphism of M, i.e., $\theta\in S$. Moreover, $A(W)(u-af)=0$, so that $u\theta-af\in W$. But then $af\neq 0$ and $af\in V$ contradicting the hypothesis that a is S-linearly independent from V. Hence $A(V)a=M$ and the theorem follows.

2. Applications

We use theorems 1 and 2 to discuss some notions and theorems extending the usual notions of primitive ring and irreducible module as well as some of the theory relating them. Let $_RM$ be a faithful R-module and let $S=\mathrm{Hom}_R(M,M)$. If for any $m\in M$, $Rm=0$ implies $m=0$, and if for any proper submodule N of $_RM$, $Nf=0$ for some nonzero $f\in S$, while also M_S is a free S-module, then R is *(left) almost primitive ring* and *$_RM$ is a characteristic R-module.*

From theorems 1 and 2 it follows via the usual argument that if R is (left) almost primitive ring, then R has an epimorphic image of the ring of all finite-by-finite matrices over S.

For the converse we have the following.

THEOREM 3. *Let S be a ring with identity such that for each proper left ideal I there is a nonzero element a of S such that $Ia=0$. Then R, the complete ring of finite-by-finite matrices over S, is a (left) almost primitive ring.*

Proof. Let E_{ij} be the matrix with (i,j) entry 1, and all other entries 0. Let $M=RE_{11}$. Then, $M=E_{11}S\oplus E_{21}S\oplus\cdots\oplus E_{n1}S\oplus\cdots$, whence M is a free right S-module as well as a left R-module. We claim that $\mathrm{Hom}_R(M,M)=S$. Thus, let $g\in\mathrm{Hom}_R(M,M)$ and let $(E_{11})g=\sum_i E_{i1}x_i$, $x_i\in S$. Therefore,

$(E_{11})g=(E_{11}\cdot E_{11})g=E_{11}(E_{11})g=E_{11}\cdot\sum_i E_{il}x_i=E_{11}x_1$. Now if $m\in M$, $m=yE_{11}$ for some $y\in R$, then $(yE_{11})g=y(E_{11})g=yE_{11}x_1$, whence $mg=mx$, for all $m\in M$, i.e., $g\in S$.

Now, let ${}_RN$ be a proper submodule of ${}_RM$. We claim that $Nf=0$ for some nonzero element f of S. Let $m=\sum_{i=1}^n E_{il}x_i$, $x_i\in S$, be an element of the proper submodule ${}_RN$ of ${}_RM$. Then, $E_{jl}m=E_{jl}x_l$, and $N=E_{11}I\oplus E_{21}I\oplus\cdots$ for some left ideal I of S. Since I is proper, $Ia=0$ for some nonzero element a of S. Hence, $Na=0$, and it suffices to take $f=a$. Clearly, $Rm=0$ implies $m=0$. Thus, R is an almost primitive ring with characteristic module ${}_RM$.

If S is a (right) Steinitz ring, i.e., a local ring with radical J such that for each sequence $\{x_i\}$ of elements of J there is an n such that $x_nx_{n-1}\cdots x_1=0$, then $Ja=0$ for some nonzero element a of S. Hence $Ia=0$ *if* I is a proper left ideal. Thus theorem 3 has the following:

COROLLARY *The complete ring of finite-by-finite matrices over a (right) Steinitz ring S is a (left) almost primitive ring.*

Collecting results we obtain:

THEOREM 4. *If S is a ring with identity such that for each proper left ideal I there is a nonzero element a of S such that $Ia=0$, then the ring R of all finite-by-finite matrices over S is (left) almost primitive.*

Conversely, if R is a (left) almost primitive ring, then R is epimorphic to a ring for which the ring of all finite-by-finite matrices over a ring S with identity, such that for each left ideal I of S there is a nonzero element a of S such that $Ia=0$, is a dense set.

Proof: Let R be (left) almost primitive and let $S=\mathrm{Hom}_R(M,M)$, where ${}_RM$ is a characteristic R-module, Let $\{m_1,m_2,\ldots\}$ be an S-basis of M. Let I be a proper left ideal of S. Then $N=\sum_i m_iI$ is a proper R-submodule of M. Hence $Nf=0$ for some nonzero $f\in S$. Thus $If=0$ and $f=a$ will do.

3. Artinian almost primitive rings

If R is Artinian and almost primitive, with characteristic module ${}_RM$, $S=\mathrm{Hom}_R(M,M)$, we let $J(R)$ be the Jacobson radical of R. Now R is isomorphic to the complete ring of $n\times n$ matrices over S for some finite number n, say $R=S_n$. Thus $J=J(R)=JS_n$, whence $J=I_n$, the Jacobson radical of S, $J(S)$, being the ideal I. Thus, $R/J=(S/I)_n$ is semisimple and Artinian, so that it is a direct sum of finitely many ideals which are simple rings, Say $(S/I)_n=S_1\oplus\cdots\oplus S_r$. Then $S_1=(I_i)_n$ for some ideal I_i of S/I, whence $(S/I)_n=(I_1)_n\oplus\cdots\oplus(I_r)_n$. Thus, $S/I=I_1\oplus\cdots\oplus I_r$ where each ideal I_i

is a division ring. It follows that:

THEOREM 5. *If R is Artinian and almost (left) primitive then $R=S_n$, the complete ring of $n\times n$ matrices, for some n, over $S=\mathrm{Hom}_R(M, M)$. Furthermore if I is the Jacobson radical of S, then $S/I=I_1\oplus\cdots\oplus I_r$, where each ideal I_r is a division ring. Also, $Ia=0$ for some nonzero element a of S.*

4. Artinian almost semisimple rings

Given a ring R, a two-sided ideal is a *(left) almost primitive ideal* if R/I is a left almost primite ring. We let $AJ(R)$ denote the intersection of all (left) almost primitive ideals. We shall say R is *(left) almost semisimple* if $AJ(R)=0$.

Suppose R is Artinian. Then $AJ(R)=I_1\cap\cdots\cap I_n$ for some finite set of almost primitive ideals. If R is Artinian (left) almost semisimple, then $AJ(R)=I_1\cap\cdots\cap I_n=0$ and there is an injection $R\rightarrow R/I_1\oplus\cdots\oplus R/I_n$, where the rings R/I_i are themselves Artinian (left) almost primitive rings, i.e., rings as described in theorem 5. Thus, R is a finite subdirect sum of (left) almost primitive rings. Under a variety of conditions this subdirect sum will be a direct sum, i.e., we have a Chinese Remainder Theorem to produce an isomorphism between R and $R/I_1\oplus\cdots\oplus R/I_n$.

If R is a subdirect sum of $R_1\oplus\cdots\oplus R_n$, where each R_i is an Artinian (left) almost primitive sing, then R is (left) almost semisimple. Indeed, $R_1=R/I_1\cap R$, where $I_1=R_2\oplus\cdots\oplus R_n$ is a (left) almost primitive ideal, so that $I_1\cap R$ is a (lefs) almost primitive ideal of R. Hence $AJ(R)=0$ and R is (left) almost semisimple. Since $R\cap R_i=R_i$, it follows that R is Artinian. Thus, we have the following:

THEOREM 6. *R is Artinian and (left) almost semisimple if and only if R is a subdirect sum of a finite number of Artinian (left) almost primitive rings.*

References

1. E. A. Behrens, *Ring theory*, Academic Press, New York, 1972.
2. B. S. Chwe and J. Neggers, *Local rings with left vanishing radical*, J. London Math. Soc. (2), **4** (1971), 374-378.
3. T. Hungerford, *Algebra*, Holt, Rinehart and Winston, New York, 1974.
4. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966.

University of Alabama